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On the $b-$adic diaphony of the Roth net and generalized Zaremba net

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In our paper we consider two classes of two-dimensional nets- the net of Roth and the generalized net of Zaremba. We give above and low estimations for the $b-$adic diaphony of these nets. We obtain an order $\mathcal{O}\left(\frac{\sqrt{\log b^\nu}}{b^\nu}\right)$ of the $b-$adic diaphony of these nets.

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1. Introduction

Let $E^s = [0,1)^s$ $(s \geq 1)$ be the s-dimensional unit cube. Let $\sigma_N = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be an arbitrary net, composed from N points in E^s. For each $\mathbf{x} = (x_1, \ldots, x_s) \in E^s$ let $A(\sigma_N; \mathbf{x}) = \{\mathbf{x}_n : \mathbf{x}_n \in \prod_{i=1}^{s}[0, x_i), 1 \leq n \leq N\}$. The quadratical discrepancy $T(\sigma_N)$ of the net σ_N is an classical measure of the irregularity of distribution and it is defined as

$$T(\sigma_N) = \left(\int_{E^s} \left|N^{-1}A(\sigma_N; \mathbf{x}) - x_1 \ldots x_s\right|^2 dx_1 \ldots dx_s\right)^{\frac{1}{2}}.$$

Roth [1] established a general lower bound of the quadratical discrepancy: For every net σ_N, composed from N points in E^s with $s \geq 2$, there is a positive constant $C(s)$, depending on the dimension s so that $T(\sigma_N) > C(s)\frac{(\log N)^{\frac{s-1}{2}}}{N}$. This lower bound is possibly the best. There exist nets Θ_N, composed from N points with $T(\Theta_N) = \mathcal{O}\left(\frac{(\log N)^{\frac{s-1}{2}}}{N}\right)$. For $s = 2$ this is the net of Zaremba [2] (see Definition 1) and for $s \geq 3$ Roth [3], [4] proposes nets, which have the best possible order of the their quadratical discrepancy.

In 1976, Zinterhof [5] proposed another measure for the irregularity of distribution, called diaphony, which is defined as

$$F(\sigma_N) = \left(\sum_{\mathbf{m} \in \mathbf{Z}^s \setminus \{\mathbf{0}\}} R^{-2}(\mathbf{m}) \left| N^{-1} \sum_{n=1}^{N} exp(2\pi \mathbf{i} < \mathbf{m}, \mathbf{x}_n >) \right|^2 \right)^{\frac{1}{2}},$$

where for $\mathbf{m} = (m_1, \ldots, m_s)$ $R(\mathbf{m}) = \prod_{i=1}^{s} max(|m_i|, 1)$, and $< \mathbf{m}, \mathbf{x}_n >$ is the inner product of the vectors $\mathbf{m}$ and $\mathbf{x}_n$ $(1 \leq n \leq N)$.

Bikovski [6] established a general lower bound of the diaphony: For every net σ_N, composed from N points in E^s with $s \geq 2$ $F(\sigma_N) > C_1(s) \frac{(\log N)^{\frac{s-1}{2}}}{N}$, where $C_1(s)$ is a positive constant, depending on the dimension s.

Grozdanov [7] obtained a class of two-dimensional nets with an order $\mathcal{O}\left(\frac{\sqrt{\log 2^\nu}}{2^\nu}\right)$ of their diaphony, which is possibly the best. Xiao [8] proved that the diaphony of the nets of Roth and Zaremba (see Definition 1) has an order $\mathcal{O}\left(\frac{\sqrt{\log b^\nu}}{b^\nu}\right)$, too.

2. Statements of the results

Let $b \geq 2$ denote a fixed integer and $\omega = exp(\frac{2\pi \mathbf{i}}{b})$. Chrestenson [9] considers a class of generalized Walsh [10] functions, defined in $b-$adic system: The Rademacher functions of order b are defined by

$$r_0(x) = \omega^k, \quad \text{if} \quad \frac{k}{b} \leq x < \frac{k+1}{b}, \quad k = 0, \ldots, b-1,$$

and for $n \geq 1$ $r_n(x) = r_0(b^n x)$.

The Chrestenson functions of order b are defined as: $w_0(x) = 1$, $\forall x \in [0,1)$ and for integer $n \geq 1$ in the form

$$n = a_1 b^{n_1} + \ldots + a_m b^{n_m},$$

where for $1 \leq j \leq m$ $a_j \in \{1, \ldots, b-1\}$ and $n_1 > \ldots > n_m \geq 0$ the $n-$th Chrestenson function is

$$w_n(x) = r_{n_1}^{a_1}(x) \ldots r_{n_m}^{a_m}(x), \quad x \in [0,1).$$

Let $\mathcal{W}(b) = \{w_n\}_{n=0}^{\infty}$ denote the system of the Chrestenson functions of order b. When $b = 2$ the original system of Walsh $\mathcal{W}(2)$ is obtained.

We note that for an arbitrary real $x \in [0,1)$ with $b-$adic development $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$ and each integer $n \geq 0$ we have the equation

$$r_n(x) = \omega^{x_n}. \tag{1}$$

For vector $\mathbf{k} = (k_1, \ldots, k_s)$ with nonnegative coordinates the Chrestenson function $w_{\mathbf{k}}(\mathbf{x})$ on E^s is $w_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} w_{k_i}(x_i)$, $\mathbf{x} = (x_1, \ldots, x_s) \in E^s$.

Using the system $\mathcal{W}(2)$ Hellekalek and Leeb [11] introduce a new version of diaphony, the so-called dyadic diaphony. The authors [12] make an extension of the definition of dyadic diaphony to the so-called b-adic diaphony. The $b-$adic diaphony of arbitrary net σ_N in E^s is defined as

$$F(\mathcal{W}(b); \sigma_N) = \left(\frac{1}{(b+1)^s - 1} \sum_{\mathbf{k} \neq \mathbf{0}} \rho(\mathbf{k}) \left| N^{-1} \sum_{n=1}^{N} w_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for the vector $\mathbf{k} = (k_1, \ldots, k_s)$ with nonnegative coordinates $\rho(\mathbf{k}) = \prod_{i=1}^{s} \rho(k_i)$ and

$$\rho(k) = \begin{cases} b^{-2g}, & \text{if } \ b^g \leq k < b^{g+1}, \quad g \geq 0, \quad g \in \mathbf{Z}, \\ 1, & \text{if } \ k = 0. \end{cases}$$

For $\gamma, \delta \in \{0, 1, \ldots, b-1\}$ we note $\gamma \oplus \delta = \gamma + \delta (mod \ \ b)$. If $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$ and $y = \sum_{j=0}^{\infty} y_j b^{-j-1}$ we put $x \oplus y = \sum_{j=0}^{\infty} (x_j \oplus y_j) b^{-j-1}$.

Let $b \geq 2$ be a fixed integer. Let $n = \sum_{j=0}^{\infty} a_j(n) b^j$ be the b-adic expansion of n and let $\phi_b(n) = \sum_{j=0}^{\infty} a_j(n) b^{-j-1}$ be the n-th element of the Van der Corput-Halton [13] sequence in base b.

Let $\nu > 0$ be an arbitrary fixed integer. Let for $0 \leq j \leq \nu - 1$ $\mu_j \equiv \alpha j + \beta (mod \ \ b)$, with fixed $\alpha, \beta \in \{0, 1, \ldots, b-1\}$ and $\mu = 0.\mu_0 \mu_1 \ldots \mu_{\nu-1}$. For $0 \leq n \leq b^\nu - 1$ we put $\psi_{b,\nu}(n) = \frac{n}{b^\nu}$ and $\phi'_b(n) = \phi_b(n) \oplus \mu$.

Let $\psi_{b,\nu} = \{\psi_{b,\nu}(n) : 0 \leq n \leq b^\nu - 1\}$ and $\phi'_{b,\nu} = \{\phi'_b(n) : 0 \leq n \leq b^\nu - 1\}$.

Definition 1. *For any positive integer ν, the Roth net and the generalized Zaremba net in base b, composed from b^ν points are defined respectively by*

$$R_{b,\nu} = \{(\psi_{b,\nu}(n), \phi_b(n)) : 0 \le n \le b^\nu - 1\},$$
$$Z_{b,\nu}^{\alpha,\beta} = \{(\psi_{b,\nu}(n), \phi'_b(n)) : 0 \le n \le b^\nu - 1\}.$$

When $\alpha = 1$ and $\beta = 0$ the net $Z_{b,\nu}^{1,0}$ is introduced by Warnock [14]. When $b = 2$ and $\alpha = 1$ and $\beta = 1$ the net $Z_{2,\nu}^{1,1}$ is the original Zaremba [2] net. If $\alpha = 0$ and $\beta = 0$ then $Z_{b,\nu}^{0,0}$ is the original net of Roth $R_{b,\nu}$.

In our paper we will obtain an order $\mathcal{O}\left(\frac{\sqrt{\log b^\nu}}{b^\nu}\right)$ of the b-adic daphony of the Roth net and the generalized Zaremba net. The next results hold.

Theorem 1. *For any integer ν the inequations hold*

$$\frac{b^2-1}{b+2}\frac{\nu}{b^{2\nu}} - \frac{b}{b+2}\frac{1}{b^{2\nu}} + \frac{b}{b+2}\frac{1}{b^{3\nu}} \le F^2(\mathcal{W}(b); R_{b,\nu}), \quad F^2(\mathcal{W}(b); Z_{b,\nu}^{\alpha,\beta})$$
$$\le \frac{b^2-1}{b+2}\frac{\nu}{b^{2\nu}} + \frac{2(b+1)}{b+2}\frac{1}{b^\nu} - \frac{2b}{b+2}\frac{1}{b^{2\nu}} + \frac{b}{b+2}\frac{1}{b^{3\nu}}\,.$$

Corollary 1. *The following equations hold*

$$\lim_{\nu\to\infty} \frac{b^\nu F(\mathcal{W}(b); R_{b,\nu})}{\sqrt{\log b^\nu}}$$
$$= \sqrt{\frac{b^2-1}{(b+2)\log b}}; \quad \lim_{\nu\to\infty} \frac{b^\nu F(\mathcal{W}(b); Z_{b,\nu}^{\alpha,\beta})}{\sqrt{\log b^\nu}} = \sqrt{\frac{b^2-1}{(b+2)\log b}}.$$

Corollary 2. *The following equations hold*

$$\lim_{\nu\to\infty} \frac{2^\nu F(\mathcal{W}(2); R_{2,\nu})}{\sqrt{\log 2^\nu}} = \frac{1}{2}\sqrt{\frac{3}{\log 2}}; \quad \lim_{\nu\to\infty} \frac{2^\nu F(\mathcal{W}(2); Z_{2,\nu})}{\sqrt{\log 2^\nu}} = \frac{1}{2}\sqrt{\frac{3}{\log 2}}\,.$$

The results, exposed in Theorem 1, Corollaries 1 an 2 have been announced in Grozdanov and Stoilova [15].

3. Preliminary results

Lemma 1. *Let* $0 \leq g_1 \leq \nu - 1$ *be an arbitrary fixed integer. An arbitrary integer* k_1, $b^{g_1} \leq k_1 < b^{g_1+1}$ *we will represent in the form*

$$k_1 = \alpha_{g_1} b^{g_1} + \alpha_{g_1-1} b^{g_1-1} + \ldots + \alpha_m b^m, \tag{2}$$

where for $0 \leq m \leq j \leq g_1$ $\alpha_j \in \{0, 1, \ldots, b-1\}$ *and* $\alpha_m, \alpha_{g_1} \neq 0$. *Using* k_1 *we define the integer*

$$k_1^* = \overline{\alpha}_m b^{\nu-1-m} + \overline{\alpha}_{m+1} b^{\nu-2-m} + \ldots + \overline{\alpha}_{g_1} b^{\nu-1-g_1},$$

where for $m \leq j \leq g_1$ $\overline{\alpha}_j \in \{0, 1, \ldots, b-1\}$ *and* $\alpha_j \oplus \overline{\alpha}_j = 0$.

Then for every integers g_2, $0 \leq g_2 \leq \nu - 1$ *and* k_2, $b^{g_2} \leq k_2 < b^{g_2+1}$ *the following equations hold*

$$\left| \sum_{i=0}^{b^\nu - 1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi_b'(i)) \right| = \begin{cases} b^\nu, & if \quad k_2 = k_1^*, \\ 0, & if \quad k_2 \neq k_1^*. \end{cases}$$

P r o o f. For an arbitrary integer i, $0 \leq i < b^\nu$ we will use the representation $i = \sum_{h=0}^{\nu-1} i_h b^h$. For an integer k_1 of the form (2) using (1) we have

$$w_{k_1}(\psi_{b,\nu}(i)) = \prod_{h=m}^{g_1} \omega^{\alpha_h i_{\nu-1-h}}. \tag{3}$$

Let $k_2 = k_1^*$. From (1) we have

$$w_{k_2}(\phi_b'(i)) = \prod_{h=m}^{g_1} \omega^{\overline{\alpha}_h (i_{\nu-1-h} + \mu_{\nu-1-h})}. \tag{4}$$

From (3) and (4) for $\forall i$, $0 \leq i < b^\nu$, $w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi_b'(i)) = \prod_{h=m}^{g_1} \omega^{\overline{\alpha}_h \mu_{\nu-1-h}}$ and

$$\left| \sum_{i=0}^{b^\nu - 1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi_b'(i)) \right| = \left| \prod_{h=m}^{g_1} \omega^{\overline{\alpha}_h \mu_{\nu-1-h}} \sum_{i=0}^{b^\nu - 1} 1 \right| = b^\nu.$$

Let now $k_2 \neq k_1^*$. In order to avoid considering different cases, we will be using the representations $k_1 = \sum_{h=0}^{\nu-1} \alpha_h b^h$, $\quad k_2 = \sum_{h=0}^{\nu-1} \beta_h b^h$, where $\alpha_h, \beta_h \in$

$\{0, 1, \ldots, b-1\}$, and for $0 \leq h \leq m-1$ and $g_1+1 \leq h \leq \nu-1$ $\alpha_h = 0$. Then from (1) we have

$$\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) = \sum_{i_0=0}^{b-1} \cdots \sum_{i_{\nu-1}=0}^{b-1} \prod_{h=0}^{\nu-1} \omega^{\alpha_h i_{\nu-1-h}} \prod_{h=0}^{\nu-1} \omega^{\beta_h(i_h+\mu_h)}$$

$$= \prod_{h=0}^{\nu-1} \omega^{\beta_h \mu_h} \prod_{h=0}^{\nu-1} \sum_{i_h=0}^{b-1} \omega^{(\alpha_{\nu-1-h}+\beta_h) i_h}. \tag{5}$$

The condition $k_2 \neq k_1^*$ shows, that there is some δ, $0 \leq \delta \leq \nu-1$, so that $\beta_\delta \neq \overline{\alpha}_{\nu-1-\delta}$. Then we have $\sum_{i_\delta=0}^{b-1} \omega^{(\alpha_{\nu-1-\delta}+\beta_\delta) i_\delta} = 0$ and from (5) we obtain

$$\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) = 0.$$

■

Lemma 2. *The following inequations hold*

$$F(\mathcal{W}(b); \psi_{b,\nu}) \leq \frac{1}{\sqrt{b^\nu}}, \quad F(\mathcal{W}(b); \phi'_{b,\nu}) \leq \frac{1}{\sqrt{b^\nu}}.$$

Proof. From the definition of the b-adic diaphony we have the next:

$$F^2(\mathcal{W}(b); \psi_{b,\nu}) = \frac{1}{b} \sum_{g=0}^{\nu-1} b^{-2g} \sum_{k=b^g}^{b^{g+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} w_k(\psi_{b,\nu}(i)) \right|^2$$

$$+ \frac{1}{b} \sum_{g=\nu}^{\infty} b^{-2g} \sum_{k=b^g}^{b^{g+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} w_k(\psi_{b,\nu}(i)) \right|^2. \tag{6}$$

We have that for each integers g, $0 \leq g \leq \nu-1$ and k, $b^g \leq k < b^{g+1}$, the functions $w_k(x)$ are situated on b^{g+1} sections, each with length b^{-g-1}. It shows that in each section there will be $b^{\nu-g-1}$ points of the net $\psi_{b,\nu}$. But the functions $w_k(x)$ take the values $\omega^0, \omega^1, \ldots, \omega^{b-1}$ and each of these values is encountered b^g times. Hence,

$$\sum_{i=0}^{b^\nu-1} w_k(\psi_{b,\nu}(i)) = b^{\nu-1} \sum_{h=0}^{b-1} \omega^h = 0. \tag{7}$$

For each integers g, $g \geq \nu$ and k, $b^g \leq k < b^{g+1}$, we will use the estimation

$$\left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} w_k(\psi_{b,\nu}(i)) \right| \leq 1. \tag{8}$$

From (6), (7) and (8) we obtain

$$F^2(\mathcal{W}(b);\psi_{b,\nu}) \leq \frac{1}{b}\sum_{g=\nu}^{\infty} b^{-2g} \sum_{k=b^g}^{b^{g+1}-1} 1 = \frac{1}{b^\nu}.$$

The demonstration of the second inequality is the same. ∎

4. Proof of Theorem 1

From the definition of the $b-$adic diaphony we have the next:

$$F^2(\mathcal{W}(b);Z_{b,\nu}^{\alpha,\beta}) = \frac{1}{b(b+2)}\sum_{\mathbf{k}\neq\mathbf{0}} \rho(\mathbf{k}) \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2$$

$$= \frac{1}{b(b+2)}\sum_{k=1}^{\infty} \rho(k) \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_k(\psi_{b,\nu}(i)) \right|^2 + \frac{1}{b(b+2)}\sum_{k=1}^{\infty} \rho(k) \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_k(\phi'_b(i)) \right|^2$$

$$+\frac{1}{b(b+2)}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \rho(k_1,k_2) \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2$$

$$= \frac{1}{b+2}F^2(\mathcal{W}(b);\psi_{b,\nu}) + \frac{1}{b+2}F^2(\mathcal{W}(b);\phi'_{b,\nu})$$

$$+\frac{1}{b(b+2)}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \rho(k_1,k_2) \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2. \quad (9)$$

For the sum of the equation (9) we have the presentation

$$\frac{1}{b(b+2)}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \rho(k_1,k_2) \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2$$

$$= \frac{1}{b(b+2)}\sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2$$

$$+\frac{1}{b(b+2)}\sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2$$

$$+\frac{1}{b(b+2)}\sum_{g_1=\nu}^{\infty} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2$$

$$+\frac{1}{b(b+2)}\sum_{g_1=\nu}^{\infty} b^{-2g_1}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=\nu}^{\infty} b^{-2g_2}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i))w_{k_2}(\phi'_b(i))\right|^2$$

$$=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4. \qquad (10)$$

We have the next estimations:

$$\Sigma_2 \le \frac{1}{b(b+2)}\sum_{g_1=0}^{\nu-1} b^{-2g_1}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=\nu}^{\infty} b^{-2g_2}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} 1 = \frac{b}{b+2}\frac{1}{b^\nu}-\frac{b}{b+2}\frac{1}{b^{2\nu}}. \qquad (11)$$

By analogy we can prove that

$$\Sigma_3 \le \frac{b}{b+2}\frac{1}{b^\nu}-\frac{b}{b+2}\frac{1}{b^{2\nu}}; \qquad (12)$$

$$\Sigma_4 \le \frac{b}{b+2}\frac{1}{b^{2\nu}}. \qquad (13)$$

For arbitrary integers m and g, $0\le m\le g$, we introduce the sets

$$A(g,m)=\{k\in\mathbf{N}: k=\sum_{j=m}^{g}\alpha_j b^j, \alpha_j\in\{0,1,\ldots,b-1\}, \alpha_m,\alpha_g\neq 0\}.$$

It is obvious that

$$|A(g,m)|=\begin{cases}(b-1)^2 b^{g-1-m}, & if \quad 0\le m\le g-1,\\ b-1, & if \quad m=g.\end{cases}$$

For the sum Σ_1 we have the next:

$$\Sigma_1=\frac{1}{b(b+2)}\sum_{g_1=0}^{\nu-1} b^{-2g_1}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=0}^{\nu-1} b^{-2g_2}$$

$$\times\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i))w_{k_2}(\phi'_b(i))\right|^2$$

$$=\frac{1}{b(b+2)}\sum_{g_1=0}^{\nu-1} b^{-2g_1}\sum_{m=0}^{g_1}\sum_{k_1\in A(g_1,m)}\sum_{g_2=0}^{\nu-1} b^{-2g_2}$$

$$\times\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i))w_{k_2}(\phi'_b(i))\right|^2$$

$$=\frac{1}{b(b+2)}\sum_{g_1=0}^{\nu-1} b^{-2g_1}\sum_{k_1\in A(g_1,g_1)}\sum_{g_2=0}^{\nu-1} b^{-2g_2}$$

$$\times \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2$$

$$+\frac{1}{b(b+2)} \sum_{g_1=1}^{\nu-1} b^{-2g_1} \sum_{m=0}^{g_1-1} \sum_{k_1\in A(g_1,m)} \sum_{g_2=0}^{\nu-1} b^{-2g_2}$$

$$\times \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2 . \tag{14}$$

From Lemma 1 for (14) we have

$$\Sigma_1 = \frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} b^{-2(\nu-1-g_1)} \sum_{k_1\in A(g_1,g_1)} 1$$

$$+\frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{m=0}^{g_1-1} b^{-2(\nu-1-m)} \sum_{k_1\in A(g_1,m)} 1$$

$$= \frac{b}{(b+2)b^{2\nu}} \sum_{g_1=0}^{\nu-1} \sum_{k_1\in A(g_1,g_1)} 1 + \frac{b}{(b+2)b^{2\nu}} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{m=0}^{g_1-1} b^{2m} \sum_{k_1\in A(g_1,m)} 1$$

$$= \frac{b(b-1)}{(b+2)} \frac{\nu}{b^{2\nu}} + \frac{(b-1)^2}{(b+2)b^{2\nu}} \sum_{g_1=0}^{\nu-1} b^{-g_1} \sum_{m=0}^{g_1-1} b^m$$

$$= \frac{b^2-1}{b+2} \frac{\nu}{b^{2\nu}} - \frac{b}{b+2} \frac{1}{b^{2\nu}} + \frac{b}{b+2} \frac{1}{b^{3\nu}}. \tag{15}$$

From (9), Lemma 2, (10), (11), (12), (13) and (15) we obtain

$$F^2(\mathcal{W}(b); Z_{b,\nu}^{\alpha,\beta}) \le \frac{b^2-1}{b+2} \frac{\nu}{b^{2\nu}} + \frac{2(b+1)}{b+2} \frac{1}{b^\nu} - \frac{2b}{b+2} \frac{1}{b^{2\nu}} + \frac{b}{b+2} \frac{1}{b^{3\nu}}.$$

It is obvious that

$$F^2(\mathcal{W}(b); Z_{b,\nu}^{\alpha,\beta}) \ge$$

$$\frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} w_{k_1}(\psi_{b,\nu}(i)) w_{k_2}(\phi'_b(i)) \right|^2 .$$

From (15) we obtain

$$F^2(\mathcal{W}(b); Z_{b,\nu}^{\alpha,\beta}) \ge \frac{b^2-1}{b+2} \frac{\nu}{b^{2\nu}} - \frac{b}{b+2} \frac{1}{b^{2\nu}} + \frac{b}{b+2} \frac{1}{b^{3\nu}},$$

so that the Theorem 1 is completely proved. ∎

Corollaries 1 and 2 are obtained directly from Theorem 1.

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