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Limit Extremal Problems

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The subgradient method of N. Z. Shor and B. T. Polyak is used for solving limit extremal problems. Under assumptions involving continuous convergence of functions strong and weak convergence of the approximations of the solution is proved.

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1. Introduction

The practice delivers sometimes extremal problems, in which the objective function is not strictly fixed. Such problems are considered as depending on some parameters (for example, the time) and are usually called **limit extremal**. Such a situation arises for the extremal problems

$$f^* = \inf\{f(x) :\ x \in X\}, \tag{1.1}$$

$$f_k^* = \inf\{f_k(x) :\ x \in X\}, \quad k = 1,\, 2,\, \dots\,, \tag{1.2}$$

where $f,\ f_k : R^m \to R$ and $X \subset R^m$. Here the function f is a limit in some sense of the functions f_k. We suppose that only the problems (1.2) are known. We search the global minima of the problem (1.1). For this purpose it is not felicitous neither to take the limit in the problems (1.2) as $k \to \infty$, nor to consider only one of the problems (1.2), [7].

The following notations will be used: $\mathrm{Conv}(R^m)$ is the set of all nonempty closed convex subset of R^m, the symbol $:=$ denotes equal by definition, $\langle x, y\rangle$ is the Euclidean scalar product of the vectors $x, y \in R^m$, $\|x\| = \sqrt{\langle x, x\rangle}$ is the norm of the vector $x \in R^m$, $B(x, r)$ is the closed ball of center x and radius r. For each positive number ϵ and $A \in \mathrm{Conv}(R^m)$ we denote the interior of A by $\mathrm{int}\, A$, the closure of A by $\operatorname{cl} A$, the diameter of the bounded set A by $\operatorname{diam} A := \sup\{\|x - y\| : x,\, y \in A\}$, the orthogonal projection of the vector x onto A by $\Pi(A, x)$, the distance between the point x and the set A by $\rho(x, A) := \inf\{\|x - y\| : y \in A\}$, the set of all points x such that $\rho(x, A) \le \epsilon$ by $S_\epsilon(A)$, the set of all points x such that $\rho(x, A) < \epsilon$ by $U_\epsilon(A)$, the restriction of the function f to A by $f|A$, the epigraph of this restriction by $\operatorname{epi}(f|A) := \{(x, \beta) \in R^{m+1} :\ \beta \ge f(x),\ x \in A\}$. For any sequence $\{y_k\}_{k=1}^\infty$ such that y_k converges to y, we denote this condition by $y_k \to y$. Denote the sets of solutions of the problems (1.1) and (1.2) by

$$X^* := \{x \in X : f(x) = f^*\} \quad \textit{and} \quad X_k^* := \{x \in X : f_k(x) = f_k^*\},$$

respectively.

Recall the following well-known concepts.

Definition 1.1. A sequence of functions $\{h_k\}_{k=1}^\infty$, $h_k\colon A \to R$, $A \subset R^m$ is said to be **continuously convergent** to some function $h\colon A \to R$ on the set A if for any sequence $\{y_k\}$, $y_k \in A$

$$y_k \to y \text{ implies that } h_k(y_k) \to h(y). \tag{1.3}$$

A sequence of functions $\{h_k\}_{k=1}^\infty$, $h_k\colon R^m \to R$, is said to be **uniformly convergent on the bounded sets** to some function $h\colon R^m \to R$, if

$$\sup\{|h_k(x) - h(x)| : x \in F\} \to 0$$

for every bounded $F \subset R^m$ [5, p.56].

It is well-known that if a sequence of continuous functions $\{h_k\}$ is uniformly convergent on a set A to a function h, then h_k continuously converges to h [11, Theorem 16.5.5]. If the set A is compact, then any continuously convergent on A to a function h sequence of continuous functions h_k is uniformly convergent [11, Theorem 16.5.5]. A sequence of continuous functions is continuously convergent if and only if this sequence is uniformly convergent on the bounded sets. Each continuously convergent sequence is pointwise convergent. Every subsequence of the continuously convergent sequence is convergent with the same limit.

Definition 1.2. The function $h\colon R^m \to R$ is said to be **quasiconvex**, if

$$h(x + t(y - x)) \le \max\{h(x), h(y)\} \text{ whenever } x,\ y \in R^m,\ 0 \le t \le 1.$$

The function $h\colon R^m \to R$ is said to be **strictly quasiconvex**, if

$$h(x+t(y-x)) < \max\{h(x), h(y)\} \text{ whenever } x,\ y \in R^m,\ h(x) \ne h(y),\ 0 \le t \le 1.$$

For any fixed point $x \in R^m$ and for arbitrary quasiconvex function $h: R^m \to R$, we consider the cone

$$\mathcal{D}\, h(x) := \{\xi \in R^m : \langle \xi, y - x\rangle \le 0 \; \forall y \in R^m \text{ such that } h(y) \le h(x)\}.$$

Really, this is the normal cone to the sublevel set of h at x.

Lemma 1.1. *[12, p.59], [9]. Let the function $h: R^m \to R$ be upper semicontinuous and quasiconvex. Then $h(y) < h(x)$ implies that $\langle \xi, y - x\rangle < 0$ for all $\xi \in \mathcal{D}\, h(x)$ such that $\xi \neq 0$.*

Lemma 1.2. *Let $X \subset R^m$ be a nonempty open convex set, and the function $h: X \to R$ be continuous. Then h is strictly quasiconvex on X iff it is quasiconvex on X, and for all $x \in X \setminus X^*$, where X^* is the set of global minimizers, there exists $\xi \in \mathcal{D}\, h(x)$ such that $\xi \neq 0$.*

Proof. Assume that the function h is strictly quasiconvex on X. It is well-known that each lower semicontinuous strictly quasiconvex function defined on a convex set is quasiconvex [10]. Therefore h is quasiconvex. Denote by $L_h(x) := \{y \in X : h(y) \le h(x)\}$ and $I_h(x) := \{y \in X : h(y) < h(x)\}$ the sublevel sets at x. It is well-known that $L_h(x) = \operatorname{cl} I_h(x)$ for all $x \in X \setminus X^*$ [8], and x is boundary point of $L_h(x)$. Then the necessity follows from Separability theorem.

Suppose that h is quasiconvex on X, and for all $x \in X \setminus X^*$, there exists $\xi \in \mathcal{D}\, h(x)$ such that $\xi \neq 0$. Assume the contrary: there exist x, $y \in X$ with $h(x) > h(y)$, and $\lambda \in (0,1)$ such that $x_\lambda := x + \lambda(y - x)$ satisfies the inequality $h(x) \le h(x_\lambda)$. Therefore, $\langle \xi, y - x\rangle \ge 0$ for all $\xi \in \mathcal{D}\, h(x_\lambda)$. Since $x_\lambda \notin X^*$ and $h(y) < h(x_\lambda)$, then, by Lemma 1.1, $\langle \xi, y - x\rangle < 0$ for all $\xi \in \mathcal{D}\, h(x_\lambda)$, $\xi \neq 0$. We got two inequalities that contradict each other. ∎

It is easy to show that if a sequence of quasiconvex functions h_k converges pointwise to a function h, then h is quasiconvex. If a sequence h_k of continuous functions is continuously convergent to a function h, then h is also continuous.

We make these comments, because in the limit extremal problems the properties of the limit functions may not be known.

We make the following common assumptions about the considered problems:

1) the set $X \in \operatorname{Conv}(R^m)$;

2) the functions f_k are real-valued, upper semicontinuous and quasiconvex;

3) $f_k \to f$ continuously on the space R^m;

4) there exist non-zero vectors in the cones $\mathcal{D}\, f_k(x)$ for all $x \in X \setminus X_k^*$, and we suppose that in the present paper are chosen such non-zero vectors from these cones.

Similar to Assumption 4) is considered in the usual subgradient method in case of quasiconvex functions [12].

By Lemma 1.2, if the functions f_k are continuous, then they are strictly quasiconvex on $\text{int}X$.

We shall investigate the convergence of the subgradient method of Shor-Polyak for solving the problem (1.1). Let the sequence $\{\lambda_k\}_{k=0}^{\infty}$ of step-size multipliers be apriori chosen such that

$$\lambda_k > 0,\ \lambda_k \to 0,\ \sum_{k=0}^{\infty} \lambda_k = \infty. \tag{1.4}$$

Suppose that x_0 is any initial approximation of the solution. Assume that for $k = 0,\, 1,\, 2, ...$ the approximate solution x_k is known. The next approximation is constructed as follows:

$$x_{k+1} = \begin{cases} \Pi(X, x_k - \lambda_k \frac{\xi_k}{\|\xi_k\|}), & \text{if } \xi_k \in \mathcal{D}\, f_k(x_k),\ \xi_k \neq 0, \\ x_k, & \text{if } \mathcal{D}\, f_k(x_k) = \{0\}. \end{cases} \tag{1.5}$$

The **subgradient method**, or so-called **generalized gradient method**, is suggested by N. Z. Shor [16] for solving the unconstrained minimization problem of a piecewise-linear convex, not necessarily differentiable function, and later applied in [17]. In [13] this method is extended to constrained minimization problems. Some of the conditions of convergence of the method are given in [6, 15]. Its further development and refinement may be seen in the books [4, 12, 14, 18].

In [7] it is studied the convergence of the subgradient method for solving limit extremal problems of the type (1.1), when X is convex and compact set, the functions f_k are convex and uniformly convergent on X. The present paper generalizes this result. In comparison to [7], now we consider the more general case when the set X is not necessary bounded, as in the usual subgradient method of Shor-Polyak, the sequence $\{f_k\}$ is continuously convergent to f being a weaker convergence than the uniform one, and the used functions are quasiconvex. The proofs are based on a different approach.

2. Weak convergence of the subgradient method

Lemma 2.1. *Let Assumption 3 be fulfilled. Then for every $a > f^*$, there exist a positive integer s, a real number $r > 0$, and $\bar{x} \in X$, for which*

$$B(\bar{x}, r) \subset I_k^a := \{x \in R^m : \ f_k(x) < a\} \,\forall\, k \geq s.$$

P r o o f. Assume that $a > f^*$ is arbitrary. By (1.1), there exists $\bar{x} \in X$ such that $f^* < f(\bar{x}) < a$. Suppose that the claim fails. Then for each integer s and for every $r_s > 0$, there exist an integer $k_s \geq s$ and $y_s \in B(\bar{x}, r_s)$ such that $y_s \notin I_{k_s}^a$. Let the sequence $\{r_s\}$ be such that $r_s \to 0$. Consequently, $y_s \to \bar{x}$, and $f_{k_s}(y_s) \geq a$ for every integer s. By Assumption 3, $f(\bar{x}) \geq a$, since each subsequence of continuous convergent sequence is continuous convergent, too. We got a contradiction. ■

Theorem 2.1. *Let Assumptions 1, 2, 3, 4 be fulfilled, and $f_k^* \to f^*$. Then, in both cases, $f^* > -\infty$ or $f^* = -\infty$ the sequence (1.5) has a subsequence $\{x_{k_s}\}_{s=1}^{\infty}$, which is* ***asymptotically minimizing*** *[19], i.e. $f_{k_s}(x_{k_s}) \to f^*$, and each accumulation point of this subsequence is solution of the problem (1.1).*

P r o o f. Assume that there exist $a > f^*$, initial approximation x_0, and an integer s such that

$$x_k \notin V_k^a := \{x \in R^m : \ f_k(x) \leq a\} \,\forall k \geq s.$$

Let us fix such an index k. Therefore, $f_k(x) \leq a < f_k(x_k)$ for all $x \in V_k^a$. It follows from Assumption 4 and Lemma 1.1 that

$$\langle \xi_k, x - x_k \rangle < 0 \,\forall k \geq s, \ \forall x \in V_k^a.$$

By Lemma 2.1 there exist $\bar{x} \in X$ and a positive number r and an integer $s_1 (s_1 \geq s)$ such that

$$\langle \xi_k, x - x_k \rangle < 0 \,\forall x \in B(\bar{x}, r), \ \forall k \geq s_1. \tag{2.1}$$

By choosing $x = \bar{x} + r\frac{\xi_k}{\|\xi_k\|}$, we obtain from (2.1) that

$$\langle \xi_k, \bar{x} - x_k \rangle < -r\|\xi_k\| \,\forall k \geq s_1. \tag{2.2}$$

Using that $\bar{x} \in X$ and the projection operator is contractive, by (2.2), we have

$$\begin{aligned} \|x_{k+1} - \bar{x}\|^2 &= \|\Pi(X, x_k - \lambda_k \tfrac{\xi_k}{\|\xi_k\|}) - \Pi(X, \bar{x})\|^2 \\ &\leq \|x_k - \lambda_k \tfrac{\xi_k}{\|\xi_k\|} - \bar{x}\|^2 = \|x_k - \bar{x}\|^2 + \lambda_k^2 \\ &+ \tfrac{2\lambda_k}{\|\xi_k\|}\langle \xi_k, \bar{x} - x_k \rangle < \|x_k - \bar{x}\|^2 + \lambda_k^2 - 2r\lambda_k. \end{aligned} \tag{2.3}$$

Since $\lambda_k \to 0$, then for all sufficiently large k we have that $\lambda_k^2 < r\lambda_k$. Whence, by (2.3), there exists an integer n $(n \geq s_1)$ such that

$$\|x_{k+1} - \bar{x}\|^2 < \|x_k - \bar{x}\|^2 - r\lambda_k \ \ \forall k \geq n.$$

By summing up these inequalities, we conclude that

$$(2.4) \qquad r\sum_{i=0}^{N-1}\lambda_{n+i} < \|x_n - \bar{x}\|^2 - \|x_{n+N} - \bar{x}\|^2 \text{ for each integer } N \geq 1,$$

which contradicts the condition $\sum_{i=0}^{\infty}\lambda_i = \infty$.

Thus, for every $a > f^*$, for any initial approximation x_0, and for each positive integer s, there exists some integer $k \geq s$ such that $x_k \in V_k^a$. Hence, there exists a subsequence $\{x_{k_s}\}$, for which $f_{k_s}(x_{k_s}) \leq a_s$, where $a_s \to f^*$, $a_s > f^*$. Then the relation $x_{k_s} \in X$ means that $f_{k_s}(x_{k_s}) \geq f^*_{k_s}$, and the claim follows from the assumptions of the theorem. ■

Remark 2.1. As we show in Section 4, the assumption $f_k^* \to f^*$ is consequence of the common assumptions in the case when the functions f_k are continuous.

3. Strong convergence of the subgradient method

For each $k = 1, 2, 3, \ldots$, if $f_k^* > -\infty$, denote

$$X_k^\delta := \{x \in R^m : \ f_k(x) \leq f_k^* + \delta\}.$$

Lemma 3.1. *Let Assumptions 1, 2, 3 be fulfilled, and X^* be nonempty and bounded. Assume that $f_k^* \to f^*$. Then for every positive number ϵ, there exist $\delta > 0$ and an integer s such that*

$$X \cap X_k^\delta \subset U_\epsilon(X^*) \, \forall \, k \geq s.$$

Proof. Assume by contradiction that there exists $\epsilon > 0$ such that for each integer s and for every $\delta_s \to 0$, $\delta_s > 0$, there exist $k_s \geq s$ and $y_s \in X \cap X_{k_s}^{\delta_s}$, for which $\rho(y_s, X^*) \geq \epsilon$.

Consider the following two cases:

1) The sequence $\{y_s\}$ is bounded. Without loss of generality $y_s \to y_0$. The set X is closed, and therefore $y_0 \in X$. From relation $y_s \in X_{k_s}^{\delta_s}$ it follows that $y_0 \in X^*$, which contrary to the inequality $\rho(y_s, X^*) \geq \epsilon$.

2) The sequence $\{y_s\}$ is unbounded. Without loss of generality $\|y_s\| \to \infty$. Let x^* be any point of the set X^*, and $d = \operatorname{diam} X^*$. Hence, $\|y_s - x^*\| > 2d$ for each integer sufficiently large s. Consider the sequence

$$z_s = x^* + \frac{2d}{\|y_s - x^*\|}(y_s - x^*).$$

Since $y_s \in X$ and $x^* \in X$, then $z_s \in X$,

$$\|z_s - x^*\| = 2d. \tag{3.1}$$

This implies that the sequence $\{z_s\}$ is bounded, and without loss of generality z_s converges to some point z_0. Since $z_s \in X$, then $z_0 \in X$. By quasiconvexity

$$f_{k_s}(z_s) \leq \max\{f_{k_s}(y_s), f_{k_s}(x^*)\} \leq \max\{f^*_{k_s} + \delta_s, f_{k_s}(x^*)\}.$$

Taking limits as $s \to \infty$, by (1.3), we get that $z_0 \in X^*$, which contradicts the equality (3.1) and the choice of d. ∎

Lemma 3.2. *Let Assumptions 1, 3 be fulfilled, the functions f_k be continuous, and X^* be nonempty and bounded. Assume that $f^*_k \to f^*$. Then for any $\delta > 0$ however small, there exist $r > 0$ and an integer s such that*

$$S_r(X^*) \subset X^\delta_k \, \forall \, k \geq s.$$

Proof. For contradiction, suppose that there exists $\delta > 0$ such that for each integer s and for every sequence of real positive numbers $\{r_s\}$, there exist a sequence of integers $\{k_s\}$, $k_s \geq s$, and a sequence $\{y_s\}$, $y_s \in S_{r_s}(X^*)$ such that $y_s \notin X^{\delta_s}_{k_s}$. Let $r_s \to 0$. The sequence $\{y_s\}$ is bounded, because X^* is bounded. Without loss of generality $y_s \to y_0$. We conclude from the inequality $f_{k_s}(y_s) > f^*_{k_s} + \delta$, by taking the limits as $s \to \infty$ that $f(y_0) \geq f^* + \delta$. This is a contradiction, since it follows from continuity of the functions f_k that f is continuous, and therefore, the set X^* is closed. Hence, $y_0 \in X^*$. ∎

Theorem 3.1. *Let Assumptions 1, 2, 3, 4 be fulfilled, the functions f_k be continuous, and $f^*_k \to f^*$. If the set X^* is nonempty and bounded, then*

$$\rho(x_k, X^*) \to 0, \text{ and } f_k(x_k) \to f^*.$$

Proof. First, we show that

$$\liminf_{k\to\infty} \rho(x_k, X^*) = 0, \tag{3.2}$$

for any initial approximation x_0. Otherwise, there exist $x_0 \in R^m$, $\epsilon > 0$ and an integer s such that $\rho(x_k, X^*) \geq \epsilon$ for all $k \geq s$. By Lemma 3.1., there exist $\delta > 0$ and an integer $s_1 \geq s$ such that $x_k \notin X^\delta_k$ for all $k \geq s_1$. By Theorem 2.1., Lemma 3.2., considering an arbitrary $x^* \in X^*$ instead of $\bar{x}$ we get a contradiction.

Second, we verify the equality

$$\lim_{k\to\infty} \rho(x_k, X^*) = 0. \tag{3.3}$$

According to the Lemmas 3.1., 3.2., and to the proof of Theorem 2.1., for each number $\epsilon > 0$ there exist an integer s and a real number $r > 0$, which satisfy the inequality

$$\langle \xi_k, x^* - x_k \rangle < -r\|\xi_k\| \; \forall x^* \in X^*, \; \forall k \geq s \text{ such that } \rho(x_k, X^*) \geq \epsilon. \tag{3.4}$$

Assume that x^* is any point of the set X^*. Consequently,

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \|\Pi(X, x_k - \lambda_k \tfrac{\xi_k}{\|\xi_k\|}) - \Pi(X, x^*)\|^2 \\ \leq \|x_k - \lambda_k \tfrac{\xi_k}{\|\xi_k\|} - x^*\|^2 &= \|x_k - x^*\|^2 + \lambda_k^2 + 2\tfrac{\lambda_k}{\|\xi_k\|}\langle \xi_k, x^* - x_k\rangle. \end{aligned}$$

If we replace in these inequalities x^* by $x_k^* := \Pi(X^*, x_k)$, then by (3.4) we get to the following inequalities

$$\begin{aligned} \rho^2(x_{k+1}, X^*) \leq \|x_{k+1} - x_k^*\|^2 &\leq \rho^2(x_k, X^*) + \lambda_k^2 - 2r\lambda_k \\ &< \rho^2(x_k, X^*) - r\lambda_k. \end{aligned} \tag{3.5}$$

for all $k \geq s$ such that $\lambda_k < r$ and $\rho(x_k, X^*) \geq \epsilon$. By (3.2), we conclude that for every $\epsilon > 0$, there exists an infinite set K of indices k, for which $\rho(x_k, X^*) < \epsilon$. Let the index $k \in K$ be such that $k \geq s$, and $\lambda_n \leq \epsilon$, $\lambda_n < r$, for each $n \geq k$. Clearly,

$$\begin{aligned} |\rho(x_{k+1}, X^*) - \rho(x_k, X^*)| &\leq \|x_{k+1} - x_k\| \\ = \|\Pi(X, x_k - \lambda_k \tfrac{\xi_k}{\|\xi_k\|}) - \Pi(X, x_k)\| &\leq \lambda_k. \end{aligned} \tag{3.6}$$

(If $x_k \in X_k^*$, then $x_{k+1} = x_k$. Otherwise, $x_{k+1} = \Pi(X, x_k - \lambda_k \frac{\xi_k}{\|\xi_k\|})$, $\xi_k \neq 0$. Hence,

$$\rho(x_{k+1}, X^*) \leq 2\epsilon. \tag{3.7}$$

If $\rho(x_{k+1}, X^*) \leq \epsilon$, then by (3.6), applied for $k+1$ instead of k,

$$\rho(x_{k+2}, X^*) \leq \rho(x_{k+1}, X^*) + \epsilon \leq 2\epsilon.$$

If $\rho(x_{k+1}, X^*) \geq \epsilon$, it follows from (3.5) that

$$\rho(x_{k+2}, X^*) \leq \sqrt{\rho^2(x_{k+1}, X^*) - r\lambda_{k+1}}.$$

Whence, by (3.7) $\rho(x_{k+2}, X^*) \leq \sqrt{4\epsilon^2 - r\lambda_{k+1}} \leq 2\epsilon$. By induction, we conclude that $\rho(x_{k+i}, X^*) \leq 2\epsilon$ for each $i = 1, 2, ...$ Since, ϵ is preassigned positive real number, however small, this implies that the equality (3.3) holds.

At last, we show that $f_k(x_k) \to f^*$. By (3.3), it follows from the assumptions of the set X^* that the sequence $\{x_k\}$ is bounded, and all its accumulation points are from X^*. According to the relation (1.3) $f_k(x_k) \to f^*$. ∎

4. Stability of the quasiconvex problem

It turn out that not all of the assumptions, considered in the theorems, are independent. Assumption $f_k^* \to f^*$ is a consequence of the rest when the functions f_k are continuous. It is connected with the so-called concepts **Hadamard well-posedness** and **stability** of the problem (1.1) (see, for example, the book [5]).

Definition 4.1. ([1]) Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of subsets of the space R^m. The set

$$\limsup_{k\to\infty} A_k := \{x \in R^m : \liminf_{k\to\infty} \rho(x, A_k) = 0\}$$

is called the **upper limit** of the sequence $\{A_k\}$, and the set

$$\liminf_{k\to\infty} A_k := \{x \in R^m : \lim_{k\to\infty} \rho(x, A_k) = 0\}$$

is called its **lower limit**.

Definition 4.2. ([2]) A sequence $\{h_k\}_{k=1}^{\infty}$ of real-valued lower semicontinuous on R^m functions is said to be **epi-convergent** to lower semicontinuous function h provided at each $x \in R^m$:

(i) whenever $\{x_k\}$ is convergent to x, then
$$h(x) \le \liminf_{k\to\infty} h_k(x_k);$$
(ii) there exists a sequence $\{y_k\}$, convergent to x such that
$$h(x) = \lim_{k\to\infty} h_k(y_k).$$

Denote the infimum of h over the set A by $v(h|A)$ and the infimum of h_k over A_k by $v(h_k|A_k)$. The following sufficient conditions for stability holds.

Theorem 4.1. *([3]) Let h, h_1, h_2,... be lower semicontinuous quasiconvex functions on R^m, and $\{h_k\}$ be epi-convergent to h. Assume that A, $A_k \in$* $\mathrm{Conv}(R^m)$ *for $k = 1, 2, 3,\ldots$. Suppose further that* $\limsup_{k\to\infty} A_k \subset A$*, and the set X^* of the global minimizer of the function h over the set A is compact, and there exists some point $x^* \in X^*$ with $(x^*, h(x^*)) \in$* $\liminf_{k\to\infty} \mathrm{epi}\,(h_k|A_k)$*. Then $v(h|A) = \lim_{k\to\infty} v(h_k|A_k)$.*

Consider the problems (1.1), (1.2). The following corollary concern to them.

Corollary 4.1. *If Assumptions 1, 2, 3 are fulfilled, the functions f_k are continuous on R^m, and the set X^* is nonempty and bounded, then $f_k^* \to f^*$.*

Proof. The sequence $\{f_k\}$ is epi-convergent to f. The function f is continuous and quasiconvex, since the functions f_k are continuous and quasiconvex. For all $x^* \in X^*$ there exists a sequence $\{y_k\}$, $y_k \in X$ such that $y_k \to x^*$. Hence, by continuous convergence, $f_k(y_k) \to f(x^*)$, and $(x^*, f(x^*)) \in \liminf_{k\to\infty} epi\,(f_k|X)$. Then the claim follows from Theorem 4.1. ∎

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