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Generalized Initial Value Problem for Singularly Perturbed Systems with Impulse Effects

L. I. Karandjulov, Y. P. Stoyanova

Presented by V. Kiryakova

A generalized Cauchy problem for linear singularly perturbed systems with generalized impulse actions is considered. The asymptotic expansion is constructed by boundary functions utilizing generalized inverse matrices.

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1. Statement of the problem

We consider a singularly perturbed system

$$(1)\quad \varepsilon\frac{dx}{dt} = Ax + \varepsilon A_1(t)x + \varphi(t),\ t \in [a,b],\ t \neq \tau_i,\ i = \overline{1,p},\ 0 < \varepsilon \ll 1,$$

$$a \equiv \tau_0 < \tau_1 < \cdots < \tau_p < \tau_{p+1} \equiv b,$$

the generalized initial condition

$$(2)\qquad Dx(a) = v$$

and the generalized impulse conditions at the fixed moments of time

$$(3)\qquad N_i x(\tau_i + 0) + M_i x(\tau_i - 0) = h_i,\ i = \overline{1,p}.$$

The coefficients of the problem $(1-3)$ satisfy the following conditions:

(H1) A is an $n \times n$ constant matrix and has eigenvalues with negative real parts,
$\lambda_i \in \sigma(A),\ \operatorname{Re}\lambda_i < 0,\ i = \overline{1,n}$.

(H2) $A_1(t)$ is an $n \times n$ matrix whose elements are continuously differentiable functions of class $C^\infty[a,b]$.

(H3) $\varphi(t) : [a,b] \to \mathbf{R}^n$ is a piecewise continuous n-dimensional vector function, which has discontinuities of the first kind in the points $\tau_i,\ i = \overline{1,p}$, *i.e.*,

$$\varphi(t) = \varphi_i(t),\ \ t \in (\tau_{i-1}, \tau_i],\ \ i = \overline{1,p+1},\ \ \varphi(a) = \varphi_1(\tau_0),\ \ \varphi(b) = \varphi_{p+1}(\tau_{p+1}),$$

$$\varphi_{i+1}(\tau_i) = \lim_{t \to \tau_i + 0} \varphi(t),\ \ i = \overline{1,p},\ \varphi_i(t) \in C^\infty(\tau_{i-1}, \tau_i],\ i = \overline{1,p+1}.$$

(H4) D is an $s \times n$ constant matrix and v is an s-dimensional constant vector.

(H5) $M_i, N_i,\ i = \overline{1,p}$, are known $k_i \times n$ constant matrices, h_i are k_i-dimensional constant vectors.

We set formally $\varepsilon = 0$ in (1). Thus we obtain the degenerate system

$$Ax + \varphi(t) = 0, \tag{4}$$

which has a solution

$$x_0(t) = -A^{-1}\varphi(t).$$

We seek an n-dimensional vector function $x(t,\varepsilon)$ piecewise continuously differentiable with respect to the first argument and continuous with respect to the second argument when $\varepsilon \in (0, \varepsilon_0]$, *i.e.*,

$$x(\cdot,\varepsilon) \in C^1([a,b]\backslash\{\tau_1,\dots,\tau_p\}),\ \ x(t,\cdot) \in C(0,\varepsilon_0],$$

satisfying the problem $(1-3)$ and the next limit relation

$$\lim_{\varepsilon \to 0} x(t,\varepsilon) = x_0(t),\ \ t \in [a,b]\backslash\{\tau_1,\dots,\tau_p\}.$$

The problem $(1-3)$ was considered in [3]. In that paper we obtained a separate requirement for each term of the asymptotic expansion, but in the present paper there is just one requirement for the definition of all elements of the asymptotic expansion of the solution of $(1-3)$. We get this by another approach, more precisely, by a modification [2] of the problem $(1-3)$.

The basic methods for research of linear and nonlinear impulse systems for ordinary differential equations are described in the monograph [5].

Singularly perturbed systems of the form

$$\frac{dx}{dt} = f(t,x,y), \quad \varepsilon\frac{dy}{dt} = g(t,x,y) \tag{5}$$

with initial condition $x(a) = v$ $(D \equiv E)$ are considered in [7].

Results from [7] are applied in [1] for the system (5) with impulse conditions of the form

$$\Delta x|_{t=\tau_i} = S_i x + a_i, \; i = \overline{1,p}. \tag{6}$$

We must note that in [5] a fundamental matrix of the solutions of the impulse system

$$\frac{dx}{dt} = A(t)x, \quad t \neq \tau_i, \quad \Delta x|_{t=\tau_i} = S_i x, \quad i = \overline{1,p},$$

under $\det(S_i + E) \neq 0$ is essential. With its help the solution of the system (5), (6) is constructed in [1].

In this paper we consider generalized impulse conditions (3) as $M_i, N_i,\; i = \overline{1,p}$, are arbitrary rectangular matrices. This shows that we cannot use the fundamental matrix from [5].

As in the paper [3] we utilize the method of boundary functions [7] and generalized inverse matrices and projectors [4], [6].

2. Asymptotic expansion

We seek a formally asymptotic expansion of the solution of the problem (1), (2), (3) in the form

$$x(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k (x_k^i(t) + \Pi_k^i(\nu_i)), \quad \nu_i = \frac{t-\tau_{i-1}}{\varepsilon}, \quad i = \overline{1,p+1}, \tag{7}$$

where $x_k^i(t)$ are the elements of the regular series and $\Pi_k^i(\nu_i)$ are boundary functions in a right neighborhood of the points $\tau_{i-1},\;\; i = \overline{1,p+1}$.

We substitute (7) in the system (1), expand the function $A_1(t)$ in Taylor series in a neighborhood of the points $t = \tau_{i-1},\;\; i = \overline{1,p+1}$, equate the coefficients at the like powers of ε, separately with respect to t and ν_i. Then for the elements of the regular series we obtain

$$x_k^i(t) = \begin{cases} -A^{-1}\varphi_i(t), & k = 0, \\ A^{-1}(Lx_{k-1}^i)(t), & k = 1, 2, \ldots, \end{cases} \tag{8}$$

where L is an operator of the form $(Lx)(t) = \frac{dx}{dt} - A_1(t)x$, $i = \overline{1,p+1}$.

The following systems are obtained for the coefficients of the singular series

$$\frac{d\Pi_k^i(\nu_i)}{d\nu_i} = A\Pi_k^i(\nu_i) + f_k^i(\nu_i), \quad k = 0, 1, 2, \ldots, \quad i = \overline{1, p+1}, \tag{9}$$

where the functions $f_k^i(\nu_i)$ have the form

$$f_k^i(\nu_i) = \begin{cases} 0, & k = 0, \\ \sum_{j=0}^{k-1} \frac{A_1^{(k-j-1)}(\tau_{i-1})}{(k-j-1)!} \nu_i^{k-j-1} \Pi_j^i(\nu_i), & k = 1, 2, \ldots. \end{cases} \tag{10}$$

We seek the solutions of the systems (9) as follows

$$\Pi_k^i(\nu_i) = X(\nu_i)c_k^i + \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_k^i(s)\, ds, i = \overline{1, p+1}, k = 0, 1, \ldots, \tag{11}$$

where c_k^i are n-dimensional unknown constant vectors and $X(\nu_i)$ is the normal fundamental matrix of the solutions of the system $\frac{dX}{dt} = AX, \;\; X(0) = E_n$.

We combine the initial condition (2) and impulse conditions (3) rewritten in the form

$$\sum_{i=1}^{p+1} l_i x^i(\cdot, \varepsilon) = \overline{h}, \tag{12}$$

where

$$l_1 x^1(\cdot, \varepsilon) = (D,\ \Theta_1,\ \ldots,\ \Theta_{p+1})^T x^1(a, \varepsilon) + (\Theta_0,\ M_1,\ \Theta_2,\ \ldots,\ \Theta_{p+1})^T x^1(\tau_1, \varepsilon),$$

$$l_i x^i(\cdot, \varepsilon) = (\Theta_0,\ \ldots,\ \Theta_{i-2},\ N_{i-1},\ \Theta_i,\ \cdots,\ \Theta_{p+1})^T x^i(\tau_{i-1}, \varepsilon) +$$

$$+ (\Theta_0,\ \ldots,\ \Theta_{i-1},\ M_i,\ \Theta_{i+1},\ \ldots,\ \Theta_{p+1})^T x^i(\tau_i, \varepsilon),\ i = \overline{2, p},$$

$$l_{p+1} x^{p+1}(\cdot, \varepsilon) = (\Theta_0,\ \ldots,\ \Theta_p,\ N_p)^T x^{p+1}(\tau_p, \varepsilon),$$

$\Theta_0,\ \Theta_i,\ i = \overline{1, p}$, are respectively $s \times n$ and $k_i \times n$, $i = \overline{1, p}$, null matrices, $\overline{h} = (v,\ h_1,\ h_2,\ \ldots,\ h_p)^T$ is a ν- dimensional vector, $\nu = s + k_1 + \cdots + k_p$.

We introduce the following notations

$D_i(\varepsilon) = l_i X\left(\frac{(\cdot)-\tau_{i-1}}{\varepsilon}\right),\ i = \overline{1, p+1}$ — $(\nu \times n)$ matrices,

$D(\varepsilon) = (D_1(\varepsilon),\ D_2(\varepsilon),\ \ldots,\ D_{p+1}(\varepsilon))$ — a $(\nu \times (p+1)n)$ matrix,

$\overline{\overline{h}}_0 = \overline{h} - \sum_{i=1}^{p+1} l_i x_0^i(\cdot)$ — a ν-dimensional vector,

(13)

$\overline{\overline{h}}_k(\varepsilon) = -\sum_{i=1}^{p+1} l_i x_k^i(\cdot) - \sum_{i=1}^{p+1} l_i \int_0^{\frac{(\cdot)-\tau_{i-1}}{\varepsilon}} X\left(\frac{(\cdot)-\tau_{i-1}}{\varepsilon}\right) X^{-1}(s) f_k^i(s)\, ds$ —

a ν-dimensional vector, $k = 1, 2, 3, \ldots,$

$\overline{c}_k = \left(c_k^1,\ c_k^2, \ldots, c_k^{p+1}\right)^T$ — a $(p+1)n$-dimensional vector, $k = 0, 1, 2, \ldots.$

We substitute the series (7) in the conditions (12) and according to (8), (9) and the notations (13) obtain the following systems

$$D(\varepsilon)\overline{c}_k = \begin{cases} \overline{\overline{h}}_0, & k = 0, \\ \overline{\overline{h}}_k(\varepsilon), & k = 1, 2, 3, \ldots. \end{cases} \tag{14}$$

We consider the case when the matrix $D(\varepsilon)$ has a representation

$$D(\varepsilon) = \overline{D} + O\left(\varepsilon^q \exp\left(-\frac{\alpha}{\varepsilon}\right)\right),$$

where $\overline{D}$ is a $\nu \times (p+1)n$ constant matrix, and by $O\left(\varepsilon^q \exp\left(-\frac{\alpha}{\varepsilon}\right)\right)$ we denote a matrix consisting of infinitely small with respect to ε elements. We neglect the exponentially small elements in the matrix $D(\varepsilon)$ and the systems (14) take the form

$$\overline{D}\overline{c}_k = \begin{cases} \overline{\overline{h}}_0, & k = 0, \\ \overline{\overline{h}}_k(\varepsilon), & k = 1, 2, 3, \ldots. \end{cases} \tag{15}$$

Let the following condition hold:

(H6) $\operatorname{rank} \overline{D} = n_1 \le \min(\nu, (p+1)n)$.

Then $\operatorname{rank} P_{\overline{D}} = (p+1)n - n_1 = q$, and $\operatorname{rank} P_{\overline{D}^*} = \nu - n_1 = d$, where $P_{\overline{D}}$ and $P_{\overline{D}^*}$ are projectors $P_{\overline{D}} : \mathbf{R}^{(p+1)n} \to \ker \overline{D}$ and $P_{\overline{D}^*} : \mathbf{R}^{\nu} \to \ker \overline{D}^*$. Thus in the $(p+1)n \times (p+1)n$ matrix $P_{\overline{D}}$ there exist q linearly independent columns and in the $\nu \times \nu$ matrix $P_{\overline{D}^*}$ there exist d linearly independent rows. We denote by $P_{\overline{D}}^q$ a matrix consisting of arbitrary q linearly independent columns of the matrix

$P_{\overline{D}}$ and by $P^d_{\overline{D}^*}$ — a matrix consisting of arbitrary d linearly independent rows of the matrix $P_{\overline{D}^*}$.

Let $k = 0$. Then from (15) we obtain the following system $\overline{D}\overline{c}_0 = \overline{\overline{h}}_0$ which has a solution

$$\overline{c}_0 = P^q_{\overline{D}}\xi_0 + \overline{D}^+\overline{\overline{h}}_0,\ \xi_0 \in \mathbf{R}^q, \tag{16}$$

if and only if

(H7) $P^d_{\overline{D}^*}\overline{\overline{h}}_0 = 0.$

Here by $\overline{D}^+$ the unique Moore - Penrose pseudoinverse matrix of the matrix $\overline{D}$ is denoted.

We substitute the solution (16) in (11) and get to

$$\Pi^i_0(\nu_i) = \overline{X}^i(\nu_i)\xi_0 + \overline{\Pi}^i_0(\nu_i),\ i = \overline{1, p+1}, \tag{17}$$

where $\overline{X}^i(\nu_i) = X(\nu_i)\left[P^q_{\overline{D}}\right]_{n_i}$, $\overline{\Pi}^i_0(\nu_i) = X(\nu_i)\left[\overline{D}^+\overline{\overline{h}}_0\right]_{n_i}$. The index n_i shows that the i-th n-tuple of rows of the matrix $P^q_{\overline{D}}$, respectively of components of the vector $\overline{D}^+\overline{\overline{h}}_0$ is taken.

We define the unknown constant vector ξ_0 from the solvability condition $P^d_{\overline{D}^*}\overline{\overline{h}}_1(\varepsilon) = 0$ of the system $\overline{D}\overline{c}_1 = \overline{\overline{h}}_1(\varepsilon)$, which is obtained from (15) for $k = 1$. In accordance with (13) the vector $\overline{\overline{h}}_1(\varepsilon)$ has the form

$$\overline{\overline{h}}_1(\varepsilon) = -\sum_{i=1}^{p+1} l_i x^i_1(\cdot) - \sum_{i=1}^{p+1} l_i \int_0^{\frac{(\cdot)-\tau_{i-1}}{\varepsilon}} X\left(\frac{(\cdot)-\tau_{i-1}}{\varepsilon}\right) X^{-1}(s) f^i_1(s)\, ds.$$

The function $f^i_1(\nu_i)$ according to (10) has the representation

$$f^i_1(\nu_i) = A_1(\tau_{i-1})\Pi^i_0(\nu_i),\ i = \overline{1, p+1}.$$

In the last equality we substitute the solutions (17). Thus

$$f^i_1(\nu_i) = A_1(\tau_{i-1})\overline{X}^i(\nu_i)\xi_0 + A_1(\tau_{i-1})\overline{\Pi}^i_0(\nu_i),\ i = \overline{1, p+1}.$$

In this way the vector $\overline{\overline{h}}_1(\varepsilon)$ has the form

$$\overline{\overline{h}}_1(\varepsilon) = \overline{\overline{h}}_{10} + B(\varepsilon)\xi_0 + b_1(\varepsilon), \tag{18}$$

where

$$
(19)\quad
\begin{aligned}
&\overline{\overline{h}}_{10} = -\sum_{i=1}^{p+1} l_i x_1^i(\cdot) \ \text{— a } \nu\text{-dimensional vector,}\\
&B(\varepsilon) = -\sum_{i=1}^{p+1} l_i \int_0^{\frac{(\cdot)-\tau_{i-1}}{\varepsilon}} X\left(\frac{(\cdot)-\tau_{i-1}}{\varepsilon}\right) X^{-1}(s) A_1(\tau_{i-1}) \overline{X}^i(s)\, ds \ \text{— a } \nu \times q \text{ matrix,}\\
&b_1(\varepsilon) = -\sum_{i=1}^{p+1} l_i \int_0^{\frac{(\cdot)-\tau_{i-1}}{\varepsilon}} X\left(\frac{(\cdot)-\tau_{i-1}}{\varepsilon}\right) X^{-1}(s) A_1(\tau_{i-1}) \overline{\Pi}_0^i(s)\, ds \ \text{— a } \nu\text{-dimensional vector.}
\end{aligned}
$$

We substitute (18) in the solvability condition $P_{D^*}^d \overline{\overline{h}}_1(\varepsilon) = 0$ and obtain

$$
Q(\varepsilon)\xi_0 = \tilde{h}_{10} + \tilde{b}_1(\varepsilon), \tag{20}
$$

where $Q(\varepsilon) = P_{D^*}^d B(\varepsilon)$ is a $d \times q$ matrix, $\tilde{h}_{10} = -P_{D^*}^d \overline{\overline{h}}_{10}$ is a d-dimensional vector,
$\tilde{b}_1(\varepsilon) = -P_{D^*}^d b_1(\varepsilon)$ is a q-dimensional vector.

The form of the matrix $B(\varepsilon)$ and the vector $b_1(\varepsilon)$ shows that they consist only of infinitely small elements with respect to ε, *i.e.*, $B(\varepsilon) = O\left(\varepsilon^s \exp\left(-\frac{\beta}{\varepsilon}\right)\right)$, $b_1(\varepsilon) = O\left(\varepsilon^q \exp\left(-\frac{\gamma}{\varepsilon}\right)\right)$, $s,\ q \in \mathbf{N}$, β, γ are positive constants. Therefore the matrix $Q(\varepsilon) = P_{D^*}^d B(\varepsilon)$ also consists only of exponentially small elements. The right-hand side of the system (20) besides the infinitely small elements $\tilde{b}_1(\varepsilon) = -P_{D^*}^d b_1(\varepsilon)$ includes the elements not depending on ε — $\tilde{h}_{10} = -P_{D^*}^d \overline{\overline{h}}_{10}$. Thus we seek the vector ξ_0 in the form of series at the negative powers of the small parameter ε

$$
\xi_0 = \xi_{00} + \xi_{01}\varepsilon^{-1} + \xi_{02}\varepsilon^{-2} + \cdots. \tag{21}
$$

Keeping in mind the representation of $\exp\left(-\frac{\alpha}{\varepsilon}\right)$ in the form

$$
\exp\left(-\frac{\alpha}{\varepsilon}\right) = e^{-F}\left(1 + \frac{q}{1!}\varepsilon^{-1} + \frac{q^2}{2!}\varepsilon^{-2} + \cdots\right) = e^{-F}\sum_{j=0}^{\infty}\frac{q^j\varepsilon^{-j}}{j!},
$$

$$
\alpha = F\varepsilon - q,\ F = -\left[-\frac{\alpha}{\varepsilon}\right],\ 0 \le \frac{q}{\varepsilon} < 1,
$$

for $B(\varepsilon)$ and $b_1(\varepsilon)$ from (19) it is easy to find

$$
B(\varepsilon) = \sum_{j=0}^{\infty} B_j\varepsilon^{-j},\ b_1(\varepsilon) = \sum_{j=0}^{\infty} b_{1j}\varepsilon^{-j},
$$

where B_j, b_{1j}, $j = 0, 1, 2, \ldots$, are known matrices.

Then the system (20) takes the form

$$\sum_{j=0}^{\infty} Q_j \varepsilon^{-j} \left(\xi_{00} + \xi_{01}\varepsilon^{-1} + \xi_{02}\varepsilon^{-2} + \cdots\right) = \sum_{j=0}^{\infty} \tilde{b}_{1j}\varepsilon^{-j}, \tag{22}$$

where

$$Q_j = P_{\overline{D}^*}^{d} B_j,\ j = 0, 1, 2, \ldots,\ \tilde{b}_{1j} = \begin{cases} -P_{\overline{D}^*}^{d}\left(\overline{\overline{h}}_{10} + b_{10}\right), & j = 0, \\ -P_{\overline{D}^*}^{d} b_{1j}, & j = 1, 2, 3, \ldots. \end{cases} \tag{23}$$

We equate the coefficients at the like powers of ε in (22) and obtain the following systems

$$Q_0\xi_{00} = \tilde{b}_{10},\ \ Q_0\xi_{0j} = \tilde{b}_{1j} - \sum_{s=1}^{j} Q_s\xi_{1,j-s},\ j = 1, 2, 3, \ldots. \tag{24}$$

Let the following condition be fulfilled:

(H8) $\operatorname{rank} Q_0 = q = d$.

From (24) we define successively ξ_{0j}, $j = 0, 1, 2, \ldots$:

$$\xi_{00} = Q_0^{-1}\tilde{b}_{10}.\ \ \xi_{0j} = Q_0^{-1}\left(\tilde{b}_{1j} - \sum_{s=1}^{j} Q_s\xi_{1,j-s}\right),\ j = 1, 2, 3, \ldots. \tag{25}$$

We substitute (21) in (17) and $\Pi_0^i(\nu_i)$ takes the form

$$\Pi_0^i(\nu_i) = \overline{X}^i(\nu_i)\sum_{j=0}^{\infty}\xi_{0j}\varepsilon^{-j} + \overline{\Pi}_0^i(\nu_i),\ i = \overline{1, p+1}, \tag{26}$$

where ξ_{0j}, $j = 0, 1, 2, \ldots$, have the representation (25).

Further we consider the systems $\overline{D}\overline{c}_k = \overline{\overline{h}}_k(\varepsilon)$, $k = 1, 2, 3, \ldots$. According to H6 the latter have solutions

$$\overline{c}_k = P_{\overline{D}}^{q}\xi_k + \overline{D}^{+}\overline{\overline{h}}_k(\varepsilon),\ \xi_k \in \mathbf{R}^q,\ k = 1, 2, 3, \ldots, \tag{27}$$

if and only if $P_{\overline{D}^*}^{d}\overline{\overline{h}}_k(\varepsilon) = 0$.

We substitute the solutions (27) in (11) and obtain

$$\Pi_k^i(\nu_i) = \overline{X}^i(\nu_i)\xi_k + \overline{\Pi}_k^i(\nu_i),\ k = 1, 2, 3, \ldots,\ i = \overline{1, p+1}, \tag{28}$$

where

$$\overline{\Pi}_k^i(\nu_i) = X(\nu_i)\left[\overline{D}^+\overline{\overline{h}}_k(\varepsilon)\right]_{n_i} + \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_k^i(s)\,ds.$$

We define the unknown vector ξ_k from the solvability condition $P_{\overline{D}^*}^d\overline{\overline{h}}_{k+1}(\varepsilon) = 0$ of the system $\overline{D}\overline{c}_k = \overline{\overline{h}}_{k+1}(\varepsilon)$. The vector $\overline{\overline{h}}_{k+1}(\varepsilon)$ has a representation analogous to the form (18) of the vector $\overline{\overline{h}}_1(\varepsilon)$:

$$\overline{\overline{h}}_{k+1}(\varepsilon) = \overline{\overline{h}}_{k+1,0} + B(\varepsilon)\xi_k + b_{k+1}(\varepsilon), \tag{29}$$

where

$$\overline{\overline{h}}_{k+1,0} = -\sum_{i=1}^{p+1} l_i x_{k+1}^i(\cdot),$$

$$b_{k+1}(\varepsilon) = -\sum_{i=1}^{p+1} l_i \int_0^{\frac{(\cdot)-\tau_{i-1}}{\varepsilon}} X\left(\frac{(\cdot)-\tau_{i-1}}{\varepsilon}\right) X^{-1}(s)\left(A_1(\tau_{i-1})\overline{\Pi}_k^i(s) + \right.$$

$$\left. + \sum_{j=0}^{k-1} \frac{A_1^{(k-j)}(\tau_{i-1})}{(k-j)!}\nu_i^{k-j}\Pi_j^i(s)\right)\,ds.$$

In accordance with (29) for ξ_k we obtain a system analogous to the system (20):

$$Q(\varepsilon)\xi_k = \tilde{h}_{k+1,0} + \tilde{b}_{k+1}(\varepsilon), \tag{30}$$

where

$$\tilde{h}_{k+1,0} = -P_{\overline{D}^*}^d\overline{\overline{h}}_{k+1,0},$$

$$\tilde{b}_{k+1}(\varepsilon) = -P_{\overline{D}^*}^d b_{k+1}(\varepsilon).$$

Then we seek ξ_k in the form

$$\xi_k = \xi_{k0} + \xi_{k1}\varepsilon^{-1} + \xi_{k2}\varepsilon^{-2} + \cdots. \tag{31}$$

Analogously to the system (24) we get

$$Q_0\xi_{k0} = \tilde{b}_{k+1,0}, \quad Q_0\xi_{kj} = \tilde{b}_{k+1,j} - \sum_{s=1}^{j} Q_s\xi_{k+11,j-s}, \; j = 1,2,3,\ldots, \tag{32}$$

where

$$\tilde{b}_{k+1,j} = \begin{cases} -P_{\overline{D}^*}^d\left(\overline{\overline{h}}_{k+1,0} + b_{k+1,0}\right), & j = 0, \\ -P_{\overline{D}^*}^d b_{k+1,j}, & j = 1,2,3,\ldots, \end{cases}$$

$$b_{k+1}(\varepsilon) = \sum_{j=0}^{\infty} b_{k+1,j}\varepsilon^{-j}.$$

The systems (32) under H8 have solutions

$$\xi_{k0} = Q_0^{-1}\tilde{b}_{k+1,0}. \ \xi_{kj} = Q_0^{-1}\left(\tilde{b}_{k+1,j} - \sum_{s=1}^{j} Q_s\xi_{1,j-s}\right), \ j = 1, 2, 3, \ldots. \tag{33}$$

We substitute (31) in (28) and obtain

$$\Pi_k^i(\nu_i) = \overline{X}^i(\nu_i)\sum_{j=0}^{\infty}\xi_{kj}\varepsilon^{-j} + \overline{\Pi}_k^i(\nu_i), \ k = 1, 2, \ldots, \ i = \overline{1, p+1}, \tag{34}$$

where ξ_{kj}, $j = 0, 1, 2, \ldots$, have the form (33).

A proof of the exponential decreasing of the boundary functions and the bound of the remainder term of the series (7) is described in [3].

Thus the following theorem is true.

Theorem 1. *Let the conditions (H1) – (H8) hold. The initial value problem with impulses (1 – 3) has a unique asymptotic solution of the form (7). The coefficients of the regular series have the representation (8) in each subinterval* $(\tau_{i-1}, \tau_i]$, $i = \overline{1, p+1}$. *The boundary functions have the form (26) for* $t \in (\tau_0, \tau_1]$ *and (34) for* $t \in (\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$, *as* ξ_{0j}, $j = 0, 1, 2, \ldots$, *have the representation (25) and* ξ_{kj}, $k = 1, 2, 3, \ldots$, $j = 0, 1, 2, \ldots$ *— the form (33). For the boundary functions the following estimate holds*

$$\|\Pi_k^i(\nu_i)\| \leq \sigma \exp(-\kappa\nu_i), \ \ i = \overline{1, p+1}, \ \ k = 0, 1, \ldots,$$

where σ *and* κ *are positive constants.*

Remark 1. If $\overline{D}$ is a nonsingular square matrix, then the problem (1), (2), (3) has a unique solution of the form (7). The coefficients of the regular series have the representation (8) and the boundary functions — the form

$$\Pi_k^i(\nu_i) = \overline{\Pi}_k^i(\nu_i), \ k = 0, 1, 2, \ldots, \ i = \overline{1, p+1},$$

as $\overline{D}^+ = \overline{D}^{-1}$.

Remark 2. If instead of the generalized impulse conditions (3) the function $x(t, \varepsilon)$ satisfies the following impulse conditions

$$M_i x(\tau_i - 0) + N_i x(\tau_i + 0) = I_i(x(\tau_i - 0)), \ \ i = \overline{1, p}, \tag{35}$$

where $M_i,\ N_i,\ i=\overline{1,p}$ satisfy H5 and $I_i(x)$ are k_i- dimensional vector functions, whose elements are continuously differentiable functions in the neighborhood of the solution of the degenerate system (4).

We substitute the series (7) in the impulse conditions (35) and expand $I_i(x(\tau_i-0))$ in Taylor series in a neighborhood of $x_0^i(\tau_i)$.

$$I_i(x(\tau_i-0))=I_i(\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\tau_i)+\Pi_k^i(\frac{\tau_i-\tau_{i-1}}{\varepsilon})))=I_i(\sum_{k=0}^{\infty}\varepsilon^k x_k^i(\tau_i))=I_i(x_0^i(\tau_i))$$

$$+I_i'(x_0^i(\tau_i))(\varepsilon x_1^i(\tau_i)+\cdots+\varepsilon^k x_k^i(\tau_i)+\cdots)+I_i''(x_0^i(\tau_i))(\varepsilon x_1^i(\tau_i)+\cdots+\varepsilon^k x_k^i(\tau_i)+\cdots)^2$$
$$+\cdots=I_i(x_0^i(\tau_i))+\varepsilon I_i'(x_0^i(\tau_i))x_1^i(\tau_i)+\varepsilon^2(I_i'(x_0^i(\tau_i))x_2^i(\tau_i)+\beta_{i2})$$
$$+\cdots+\varepsilon^k(I_i'(x_0^i(\tau_i))x_k^i(\tau_i)+\beta_{ik})+\cdots=h_0^i+\varepsilon h_1^i+\varepsilon^2 h_2^i+\cdots,$$

where β_{ik} are expressed by $x_\nu^i(\tau_i),\ \nu=\overline{1,k-1}$.

In this case

$$\overline{\overline{h}}_0=-\sum_{i=1}^{p+1}h_0^i-\sum_{i=1}^{p+1}l_i x_0^i(\cdot),$$

$$\overline{\overline{h}}_k(\varepsilon)=-\sum_{i=1}^{p+1}h_k^i-\sum_{i=1}^{p+1}l_i x_k^i(\cdot)-\sum_{i=1}^{p+1}l_i\int_0^{\frac{(\cdot)-\tau_{i-1}}{\varepsilon}}X\left(\frac{(\cdot)-\tau_{i-1}}{\varepsilon}\right)X^{-1}(s)f_k^i(s)ds,$$
$$k=1,2,3,\ldots.$$

The form of the latter does not lead to essential changes in the algorithm above. This shows that in this case we also may utilize this approach.

3. Example

Let the problem (1–3) have the following coefficients $A=\begin{pmatrix}1&-4\\2&-5\end{pmatrix}$, $A_1(t)=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $\varphi(t)=\begin{cases}(1-t,\ \ t)^T, & t\in[0,1],\\ 0 & t\in(1,2],\end{cases}$ $D=(1,\ \ 2)$, $v=4$, $N_1=\begin{pmatrix}0&0\\1&3\\2&1\end{pmatrix}$, $M_1=\begin{pmatrix}-1&2\\0&0\\1&-2\end{pmatrix}$, $h_1=\begin{pmatrix}2/3\\1\\1/3\end{pmatrix}$.

Then

$$X(t)=\begin{pmatrix}2e^{-t}-e^{-3t} & 2e^{-3t}-2e^{-t}\\ e^{-t}-e^{-3t} & 2e^{-3t}-e^{-t}\end{pmatrix},\ X^{-1}(t)=\begin{pmatrix}2e^{t}-e^{3t} & 2e^{3t}-2e^{t}\\ e^{t}-e^{3t} & 2e^{3t}-e^{t}\end{pmatrix}.$$

According to (8) for the elements of the regular series we find $x_0^1(t)=\frac{1}{3}\begin{pmatrix}5-9t\\2-3t\end{pmatrix}$, $x_1^1(t)=\frac{1}{3}\begin{pmatrix}20-16t\\9-7t\end{pmatrix}$, $t\in[0,1]$, $x_0^2(t)=x_1^2(t)=0$,

$t \in (1, 2]$.
Further

$$\overline{D} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \ \overline{D}^{+} = \frac{1}{5}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -1 \end{pmatrix}, \ P_{\overline{D}}^{1} = \frac{1}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$P_{\overline{D}^*}^{1} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, $\overline{\overline{h}}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and the condition (H7) $P_{\overline{D}^*}^{1}\overline{\overline{h}} = 0$ holds.

For c_0^1 and c_0^2 in accordance with (16) we obtain

$$c_0^1 = \frac{1}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\xi_0 + \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ c_0^2 = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \xi_0 \in \mathbf{R}.$$

For the definition of ξ_0 from (21) we have the system

$$\frac{3}{10}e^{-\frac{1}{\varepsilon}}\left(e^{-\frac{2}{\varepsilon}} - 1\right)\xi_0 = \frac{1}{10}e^{-\frac{1}{\varepsilon}}\left(1 - e^{-\frac{2}{\varepsilon}}\right).$$

From the last system we determine $\xi_0 = -\frac{1}{3}$.

Further for c_1^1 and c_1^2 we get

$$c_1^1 = \frac{1}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\xi_1 - \frac{38}{15}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ c_1^2 = 0, \ \xi_1 \in \mathbf{R}.$$

The system for ξ_1 according to (30) has the form

$$-\frac{3}{10}e^{-\frac{1}{\varepsilon}}\left(e^{-\frac{2}{\varepsilon}} - 1\right)\xi_1 = -\frac{4}{9} + \frac{81}{60}e^{-\frac{1}{\varepsilon}} + \frac{1}{6}\frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon} - \frac{81}{60}e^{-\frac{3}{\varepsilon}}.$$

We neglect the exponentially small elements of the form $O\left(e^{-\frac{3}{\varepsilon}}\right)$ and $O\left(\frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon}\right)$ and the last system takes the form

$$\frac{3}{10}e^{-\frac{1}{\varepsilon}}\xi_1 = -\frac{4}{9} + \frac{81}{60}e^{-\frac{1}{\varepsilon}}.$$

Keeping in mind that $e^{-\frac{1}{\varepsilon}} = e^{-F}\sum_{j=0}^{\infty}\frac{q^j}{j!}\varepsilon^{-j}$ (for instance, if $\varepsilon = 0,012$, then $F = 84$, $q = 0,008$ and $\frac{q}{\varepsilon} = \frac{2}{3} < 1$), from (31) and (33) we obtain

$$\xi_1 = \frac{9}{2} - \frac{40}{27}e^{F} + \varepsilon^{-1}\frac{40}{27}qe^{F} - \varepsilon^{-2}\frac{20}{27}q^2e^{F} + \cdots.$$

Thus

$$x^1(t,\varepsilon)=\frac{1}{3}\begin{pmatrix}5-9t\\2-3t\end{pmatrix}+\frac{e^{-3\frac{t}{\varepsilon}}}{3}\begin{pmatrix}1\\1\end{pmatrix}+\varepsilon\frac{1}{3}\begin{pmatrix}20-16t\\9-7t\end{pmatrix}+$$

$$+\varepsilon\left(\frac{e^{-\frac{t}{\varepsilon}}}{5}\begin{pmatrix}4e^{-2\frac{t}{\varepsilon}}-6\\4e^{-2\frac{t}{\varepsilon}}-3\end{pmatrix}\left(\frac{9}{2}-\frac{40}{27}e^F+\varepsilon^{-1}\frac{40}{27}qe^F-\varepsilon^{-2}\frac{20}{27}q^2e^F+\cdots\right)+\right.$$

$$\left.+\frac{e^{-\frac{t}{\varepsilon}}}{30}\begin{pmatrix}162-238e^{-2\frac{t}{\varepsilon}}\\81-233e^{-2\frac{t}{\varepsilon}}\end{pmatrix}\right)+O(\varepsilon^2),\ t\in[0,1],$$

$$x^2(t,\varepsilon)=\frac{e^{-\frac{t-1}{\varepsilon}}}{5}\begin{pmatrix}2\\1\end{pmatrix}+\varepsilon\frac{e^{-\frac{t-1}{\varepsilon}}}{10}\begin{pmatrix}8\frac{t-1}{\varepsilon}+e^{-2\frac{t-1}{\varepsilon}}-1\\4\frac{t-1}{\varepsilon}+e^{-2\frac{t-1}{\varepsilon}}-1\end{pmatrix}+O(\varepsilon^2),\ t\in(1,2].$$

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Dept. of Applied Mathematics and Computer Science
Technical University of Sofia
P.O. Box 384, Sofia 1000, BULGARIA

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