Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New Series Vol. 17, 2003, Fasc. 1-2

On the Semiprimitivity of Crossed Products

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Presented by Bl. Sendov

Let $K * G$ be any crossed product of a multiplicative group G over an associative ring K and let $\mathcal{J}(K * G)$ be the Jacobson radical of $K * G$. If K is an algebra over a field of characteristic zero, then we prove that there exists a normal subgroup H of G such that $H' = H$ and $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$. Furthermore, if H is a subnormal subgroup of G and all factors of the subnormal series are either locally finite, or generalized solvable, then again $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$. Moreover, if K is a ring of finite characteristic $m > 0$, G is a group with no p-elements for every prime divisor p of m and G has a subnormal series with factors which are either locally finite, or locally solvable, then $\mathcal{J}(K * G) \subseteq \mathcal{B}(K)(K * G)$ where $\mathcal{B}(K)$ is the Brown-McCoy radical of K.

MR Subject Classification: 16S34, 20C07

Key words and phrases: crossed product, twisted group ring, radical at Jacobson

1. Introduction

Let $K * G$ be a crossed product [2] of the multiplicative group G over the associative ring K with respect to the factor set $\rho = \{\rho(g,h) \in K^* \mid g,h \in G\}$ and the mapping $\sigma : G \to AutK$, where K^* is the multiplicative group of K and $AutK$ is the group of the automorphisms of K. This implies that $K * G$ is simultaneously an associative ring and a free K-module with a basis $\overline{G} = \{\bar{g} \in K * G \mid g \in G\}$, where the elements of $\overline{G}$ satisfies the conditions

$$\bar{g}\bar{h} = \overline{gh}\rho(g,h), \qquad \alpha\bar{g} = \bar{g}\alpha^{g^\sigma} \quad (g,h \in G,\ \alpha \in K)$$

[1]The first author's research was supported by the Science Fund of Bourgas University "A. Zlatarov" under contract NI-27/2000.

[2]The second author's research was supported by PU-25-MM.

and $\alpha^{g\sigma}$ is the image of $\alpha \in K$ under the action of the automorphism $g\sigma \in Aut\,K$. Since $\bar{f}(\bar{g}\bar{h}) = (\bar{f}\bar{g})\bar{h}$ and $(\alpha\bar{g})\bar{h} = \alpha(\bar{g}\bar{h})$, we have the following conditions for associativity

$$\rho(f, gh)\rho(g, h) = \rho(fg, h)\rho(f, g)^{h\sigma}, \quad \alpha^{g\sigma.h\sigma} = \rho(g, h)^{-1}\alpha^{(gh)\sigma}\rho(g, h)$$

for all $f, g, h \in G$ and $\alpha \in K$.

We shall assume that the basis of $K * G$ is normalized [8], i.e. $\bar{1}$ $(1 \in G)$ is the identity element of $K * G$. Then $\rho(g, 1) = \rho(1, g) = 1$ $(g \in G)$ and 1σ is the identity automorphism of K (see [2]).

We shall say that the crossed product $K * G$ is semiprimitive if its Jacobson radical $\mathcal{J}(K * G)$ is zero. The semiprimitivity problem for crossed products of infinite groups has been studied at first by A. A. Bovdi [2] and A. E. Zalesskii [14]. It is clear that this problem generalizes the corresponding semiprimitivity problem for group rings, which was investigated by many authors (see [8,9,10]). Here we use some methods from [8,12,13] for group rings and we prove several theorems which generalize some well known results for group rings.

Let K be an algebra over the field Q of the rational numbers and let G be any group. Then we prove that there is a normal subgroup H of the group G such that the commutator subgroup $H' = [H, H]$ coincides with H and $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$. Furthermore, let M be the set of all locally finite, locally nilpotent, locally free and generalized solvable groups. If H is a subnormal subgroup of G such that all factors of the subnormal series are M-groups and K is a Q-algebra, then again $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$. Finally, let K be a ring of finite characteristic $m > 0$ and let G be a group with no p-elements for every prime divisor p of m. If G has a subnormal series with either locally finite, or locally solvable factors and $K_\rho G$ is a twisted group ring of G over K, then $\mathcal{J}(K_\rho G) \subseteq \mathcal{B}(K)(K * G)$. Here $\mathcal{B}(K)$ is the Brown-McCoy radical of K, that is $\mathcal{B}(K)$ is the intersection of the maximal ideals of K [1].

2. Preliminary results

Further let $K * G$ denote a crossed product of a multiplicative group G over an associative ring K. In this section we collect some facts concerning the Jacobson radical $\mathcal{J}(K * G)$ of $K * G$ and provide the auxiliary results that are need in this paper.

Lemma 2.1. *If H is a normal subgroup of G with a finite index n, then*

$$\mathcal{J}(K * G)^n \subseteq \mathcal{J}(K * H)(K * G) \subseteq \mathcal{J}(K * G).$$

Furthermore, if $n \in K^*$, then $\mathcal{J}(K * G) = \mathcal{J}(K * H)(K * G)$.

Proof. The statement is proved in [9, Theorem 4.2] when H is the identity subgroup of G. If $H \neq \{1\}$, then $K * G = (K * H) * G/H$ is a crossed product of the finite group G/H over the ring $K * H$ [4] and again in view of [9, Theorem 4.2] the result follows. ∎

This result has a number of extremely useful consequences, some of which we offer below.

For any subgroup H of the group G there is a natural projection $\pi_H : K * G \to K * H$, given by

$$\pi_H\Big(\sum_{g\in G} \bar{g}\alpha_g\Big) = \sum_{g\in H} \bar{g}\alpha_g.$$

It is clear that $\pi_H(u + v) = \pi_H(u) + \pi_H(v)$ and $\pi_H(au) = a\pi_H(u)$, $\pi_H(ua) = \pi_H(u)a$ for $u, v \in K * G$ and $a \in K * H$. Moreover, if H is a normal subgroup of G and $u \in K * G$, $g \in G$, then $\pi_H(\bar{g}^{-1}u\bar{g}) = \bar{g}^{-1}\pi_H(u)\bar{g}$.

The following lemma is proved in [7] (see also [8, Lemma 1.1.6] and [8, Lemma 7.1.5]).

Lemma 2.2. *Let H be a subgroup of G and let A be a right ideal of $K * H$. Then*

(i). The right ideal $B = A(K*G)$ of the ring $K*G$ satisfies the condition

$$A = \pi_H(B) = B \cap (K * H);$$

(ii). $\mathcal{J}(K * G) \cap (K * H) \subseteq \mathcal{J}(K * H)$.

Now we shall prove

Lemma 2.3. *(i). If $\mathcal{J}(K*L) \subseteq \mathcal{J}(K)(K*L)$ for every finitely generated subgroup L of the group G, then $\mathcal{J}(K * G) \subseteq \mathcal{J}(K)(K * G)$;*

(ii). If $\mathcal{J}(K)(K * L) \subseteq \mathcal{J}(K * L)$ for every finitely generated subgroup L of the group G, then $\mathcal{J}(K)(K * G) \subseteq \mathcal{J}(K * G)$.

Proof. (i). If $a \in \mathcal{J}(K*G)$ and $L = \langle Supp\, a\rangle$ is the supporting subgroup of a, then L is a finitely generated subgroup of G and by the condition it follows that $\mathcal{J}(K * L) \subseteq \mathcal{J}(K)(K * L)$. Thus, in view of Lemma 2 (ii) we obtain

$$a \in \mathcal{J}(K * G) \cap (K * L) \subseteq \mathcal{J}(K * L) \subseteq \mathcal{J}(K)(K * L) \subseteq \mathcal{J}(K)(K * G).$$

Therefore, $\mathcal{J}(K * G) \subseteq \mathcal{J}(K)(K * G)$ and (i) is proved.

(ii). Suppose that $\alpha \in \mathcal{J}(K)$ and $x = \alpha a$ $(a \in K * G)$ is any element of the right ideal $I = \alpha(K * G)$ of the ring $K * G$. If $L = \langle Supp\, a \rangle$, then L is finitely generated and $\mathcal{J}(K)(K * L) \subseteq \mathcal{J}(K * L)$. Since $\alpha \in \mathcal{J}(K)$ and $a \in K * L$, this shows that $x = \alpha a \in \mathcal{J}(K * L)$. Therefore the element x is quasi-invertible and thus we conclude that I is a quasi-regular ideal of $K * G$. This implies that $\mathcal{J}(K)(K * G) \subseteq \mathcal{J}(K * G)$ and the lemma is proved. ∎

Now we shall consider a class of crossed products $K * G$ where G has some special ascending or descending series of subgroups.

Let H be a subgroup of the group G and let Ω be a totaly ordered set. We shall say that a set $S(H) = \{(U_i, V_i) \mid i \in \Omega\}$ of pairs (U_i, V_i) of subgroups of G is an ascending (resp. descending) H-series of G if

(i). $H \leq U_i \lhd V_i$ for all $i \in \Omega$;

(ii). $V_i \leq U_j$ for all $i < j$ (resp. $i > j$);

(iii). $G \setminus H = \bigcup\limits_{i \in \Omega} (V_i \setminus U_i)$.

This is, of course, a generalization of the different ascending (resp. descending) series associated with the group G (see [8, p. 293]). We shall say that H is an S-subnormal subgroup of G if G has at least one H-series $S(H)$. Further, let M be some class of groups. Then H is said to be an SM-subnormal subgroup of G if all factor groups V_i/U_i $(i \in \Omega)$ of the series $S(H)$ are M-groups. Moreover, G is an SM-group, if the identity subgroup is SM-subnormal in G.

Lemma 2.4. *Let H be a subgroup of the group G and let $S(H) = \{(U_i, V_i) \mid i \in \Omega\}$ be any series of G. If $\mathcal{J}(K * V_i) \subseteq \mathcal{J}(K * U_i)(K * V_i)$ for all $i \in \Omega$, then $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$.*

Proof. If $T(G/H)$ is some transverzal of H in G and $0 \neq a \in \mathcal{J}(K * G)$, then

$$a = a_1\bar{g}_1 + a_2\bar{g}_2 + \cdots + a_n\bar{g}_n$$

where $0 \neq a_i \in K * H$ and $g_i \in T(G/H)$ for $i = 1, 2, \ldots, n$. It suffices to show that $a_i \in \mathcal{J}(K * H)$ for all $i = 1, 2, \ldots, n$.

First by induction on the $T(G/H)$-support size $\|a\| = n$ we shall prove that if $a \in \mathcal{J}(K * U_i)$ for some $i \in \Omega$ and $\pi_H(a) \neq 0$, then $\pi_H(a) \in \mathcal{J}(K * H)$. Here π_H is the natural projection on $K * H$. Really, if $\|a\| = 1$, $\pi_H(a) \neq 0$ and $a \in \mathcal{J}(K * U_i)$, then $a = a_1\bar{g}_1$ $(g_1 \in H)$ and by Lemma 2.2 (ii),

$$\pi_H(a) = a_1\bar{g}_1 \in \mathcal{J}(K * U_i) \cap (K * H) \subseteq \mathcal{J}(K * H).$$

Let us assume that $\pi_H(a) \in \mathcal{J}(K * H)$ for every $a \in \mathcal{J}(K * G)$ when $a \in \mathcal{J}(K * U_i)$ for some $i \in \Omega$ and $\|a\| \leq k - 1$ and let $\|a\| = k > 1$. Since $S(H)$ is a series of

G and $\|a\|$ is finite, we conclude that there exists a pair $(U_j, V_j) \in S(H)$ such that $a \in K * V_j$ and $a \notin K * U_j$. Then Lemma 2.2 (ii) implies that

$$a \in \mathcal{J}(K * G) \cap (K * V_j) \subseteq \mathcal{J}(K * V_j).$$

Hence, by the condition of the lemma, we obtain

$$a \in I = \mathcal{J}(K * U_j)(K * V_j).$$

Now Lemma 2.2 (i) yields

$$\mathcal{J}(K * U_j) = \pi_{U_j}(I) = I \cap (K * U_j),$$

where π_{U_j} is the natural projection of $K * U_j$. This shows that $\pi_{U_j}(a) \in \mathcal{J}(K * U_j)$. It is clear that $\|\pi_{U_j}(a)\| \leq \|a\|$, because $a \notin K * U_j$. Then the induction assumption gives

$$\pi_H(a) = \pi_H(\pi_{U_j}(a)) \in \mathcal{J}(K * H).$$

Thus we obtain that $\pi_H(a) \in \mathcal{J}(K * H)$ for all $a \in \mathcal{J}(K * G) \cap \mathcal{J}(K * U_i)$ for some $i \in \Omega$.

Now $b = a\bar{g}_i^{-1}$ $(1 \leq i \leq n)$ is also an element of $\mathcal{J}(K*G)$ with $\pi_H(b) = a_i$. If $b \in K * V_j$ $(j \in \Omega)$, then

$$b \in \mathcal{J}(K * G) \cap (K * V_j) \subseteq \mathcal{J}(K * V_j) \subseteq \mathcal{J}(K * U_j)(K * V_j),$$

according to Lemma 2.2 (ii) and the condition of the lemma. Therefore, by Lemma 2.2 (i) as above we obtain $\pi_{U_j}(b) \in \mathcal{J}(K * U_j)$ and hence

$$\pi_H(\pi_{U_j}(b)) = \pi_H(b) = a_i \in \mathcal{J}(K * H)$$

for all $i = 1, 2, \ldots, n$, as was to be shown. ∎

We close this section with the following lemma which is proved in [11, Corollary 2 and Corollary 6].

Lemma 2.5. *Let K be a field of characteristic $p \geq 0$. If G is a group with no elements of order p when $p > 0$, then every twisted group ring $K_\rho G$ has no non-zero nil ideals.*

3. Radicals of crossed products

In this section we shall prove the main results of this paper.

Proposition 3.1. *Let H be a normal subgroup of G with a locally finite factor group G/H. Then* *(i).* $\mathcal{J}(K*H)(K*G) \subseteq \mathcal{J}(K*G)$;

(ii). If the order of every element of G/H is invertible in K, then $\mathcal{J}(K*G) = \mathcal{J}(K*H)(K*G)$.

Proof. Suppose first that $H = \{1\}$. Then G is a locally finite group and every finitely generated subgroup L of G is finite. Thus by Lemma 2.1 with $H = \{1\}$ we obtain that $\mathcal{J}(K)(K*L) \subseteq \mathcal{J}(K*L)$. Now Lemma 2.3 (ii) shows that $\mathcal{J}(K)(K*G) \subseteq \mathcal{J}(K*G)$, i.e. the assertion (i) is proved when $H = \{1\}$. If every element of G has an invertible order in K, then $|L| \in K^*$ and Lemma 2.1 yields the inclusion $\mathcal{J}(K*L) \subseteq \mathcal{J}(K)(K*L)$. Hence by Lemma 2.3 (i) we get that $\mathcal{J}(K*G) \subseteq \mathcal{J}(K)(K*G)$. Therefore (ii) is also proved when $H = \{1\}$.

In general, suppose that $H \neq \{1\}$. Then $K*G = (K*H)*G/H$ is a crossed product of G/H over $K*H$ [4]. Since $K^* \subseteq (K*H)^*$, from what has been proved above we obtain

$$\mathcal{J}(K*H)(K*G) = \mathcal{J}(K*H)((K*H)*G/H) \subseteq \mathcal{J}((K*H)*G/H) = \mathcal{J}(K*G),$$

i.e. we obtain the assertion (i). In a similar manner we prove and the assertion (ii). ■

Using the terminology of Passman [8, p. 304], we say that the subgroup H of the group G controls the ideal I of the crossed product $K*G$ if $I = (I \cap (K*H))(K*G)$. It is known [7, Lemma 2.6] that if the subgroups H_i $(i \in \Omega)$ control the ideal I, then their intersection $H = \bigcap_{i\in\Omega} H_i$ also controls I (for group rings see [8, Lemma 8.1.1]).

Proposition 3.2. *Let K be any ring and at least one prime number p is invertible in K. Let H be a normal subgroup of G with an abelian factor group G/H.*

(i). If G/H is a free abelian group, then

$$\mathcal{J}(K*G) = (\mathcal{J}(K*G) \cap K*H)(K*G);$$

(ii). If the order of every torsion element of G/H is invertible in K, then

$$\mathcal{J}(K*G) \subseteq \mathcal{J}(K*H)(K*G).$$

Proof. Suppose first that $H = \{1\}$.

(i). Let G be a free abelian group and let $G = \prod_{k\geq 1} \langle g_k \rangle$ be a decomposition of G as a direct product of infinite cyclic groups [6, p. 118]. If the prime number p is invertible in K, then we set $G_n = \prod_{k\geq 1}\langle g_k^{p^n}\rangle$ for every natural number $n \geq 1$. So $G = G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$ is an infinite strongly descending series of G such that all factor groups G_i/G_{i+1} $(i \geq 0)$ are locally finite p-groups. Since $p^{-1} \in K^*$, in view of Proposition 3.1 we obtain $\mathcal{J}(K * G_i) = \mathcal{J}(K * G_{i+1})(K * G_i)$ for all $i \geq 0$. Now with induction on n we conclude that $\mathcal{J}(K * G) = \mathcal{J}(K * G_n)(K * G)$ for all $n \geq 0$. Hence by Lemma 2.2 (i) we have

$$\mathcal{J}(K * G) = (\mathcal{J}(K * G) \cap (K * G_n))(K * G),$$

i.e. G_n controls $\mathcal{J}(K * G)$ for all $n \geq 0$. Then the intersection $S = \bigcap_{n\geq 0} G_n$ also controls the ideal $\mathcal{J}(K * G)$ [7, Lemma 2.6]. But it is easy to see that $S = \{1\}$ and therefore

$$\mathcal{J}(K * G) = (\mathcal{J}(K * G) \cap K)(K * G),$$

i.e. (i) is proved when $H = \{1\}$.

(ii). Let G be a torsion free abelian group. If L is a finitely generated subgroup of G, then L is a free abelian group and in view of (i) and Lemma 2.2 (ii) we have

$$\mathcal{J}(K * L) = (\mathcal{J}(K * L) \cap K)(K * L) \subseteq \mathcal{J}(K)(K * L).$$

Then Lemma 2.3 (i) implies $\mathcal{J}(K * G) \subseteq \mathcal{J}(K)(K * G)$.

If G is not torsion free and G_0 is the torsion subgroup of G, then $K * G = (K * G_0) * G/G_0$ where G/G_0 is a torsion free group. From this we see that

$$\mathcal{J}(K * G) \subseteq \mathcal{J}(K * G_0)[(K * G_0) * (G/G_0)] = \mathcal{J}(K * G_0)(K * G).$$

Now, according to Proposition 3.1, we have $\mathcal{J}(K * G_0) \subseteq \mathcal{J}(K)(K * G_0)$ and thus $\mathcal{J}(K * G) \subseteq \mathcal{J}(K)(K * G)$.

In general, if $H \neq \{1\}$, the result follows as above from the crossed product $K * G = (K * H) * G/H$ and the proof is completed. ∎

Corollary 3.3. *Let H be a subgroup of G and let $S(H) = \{(U_i, V_i) | i \in \Omega\}$ be an H-series of G. If all factor groups V_i/U_i $(i \in \Omega)$ are abelian and K is a Q-algebra, then $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$.*

Indeed, since the order of every torsion element of V_i/U_i $(i \in \Omega)$ is invertible in K, by the preceding proposition we have $\mathcal{J}(K * V_i) \subseteq \mathcal{J}(K * U_i)(K * V_i)$ for all $i \in \Omega$. Then the statement follows from Lemma 2.4.

Corollary 3.3 shows that if G is an SN-group (in [5] and [6] G is RN-group, i.e. G is a generalized solvable group) and K is a semiprimitive Q-algebra, then $\mathcal{J}(K * G) = O$ for every crossed product $K * G$. This result generalizes some well known results for group rings (see [12,13]).

Proposition 3.2 (ii) can be generalized for locally nilpotent group [5,6]. Indeed, we have the following

Theorem 3.4. *Let K be any ring and at least one prime number p is invertible in K. If H is a normal subgroup of G with locally nilpotent factor group G/H and the order of every torsion element of G/H is invertible in K, then $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$.*

Proof. Suppose first that $H = \{1\}$ and let L be any finitely generated subgroup of G. Then L is a nilpotent group and the torsion part L_0 of L is a finite normal subgroup of L [5, Theorem 16.2.7]. Moreover, the factor group L/L_0 has a upper central series with torsion free abelian factors [5, Theorem 17.2.2]. Hence L has a finite normal series $L_0 \subset L_1 \subset \cdots \subset L_n = L$ with torsion free factors L_i/L_{i-1} for $i = 1, 2, \ldots, n$. Now we consider the ascending series $S(L_0) = \{(U_i, V_i) \,|\, i = 1, 2, \ldots, n\}$, where $U_i = L_{i-1}$ and $V_i = L_i$ $(1 \le i \le n)$. By Proposition 3.2 (ii) and Lemma 2.4 we receive $\mathcal{J}(K*L) \subseteq \mathcal{J}(K*L_0)(K*L)$. But $|L_0| \in K^*$ and by Lemma 2.1 it follows that $\mathcal{J}(K * L_0) \subseteq \mathcal{J}(K)(K * L_0)$. Thus $\mathcal{J}(K * L) \subseteq \mathcal{J}(K)(K * L)$ and Lemma 2.3 (i) yields $\mathcal{J}(K * G) \subseteq \mathcal{J}(K)(K * G)$.

If $H \neq \{1\}$, then $K * G = (K * H) * G/H$ and the result again follows. ∎

If H and L are subgroups of G, then by $[H, L]$ we denote the subgroup of G which is generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ with $x \in H$ and $y \in L$. Then $G' = [G, G]$ is the commutator subgroup of G.

Recall that G is a locally free group if every finitely generated subgroup of G is free [6, p. 239]. For each free group G we have $G \neq G'$, but there exist locally free groups with the property $G = G'$ [6, p. 239].

Theorem 3.5. *If the factor group G/H is locally free and at least one prime number p is an invertible element in the ring K, then $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$.*

Proof. As was seen in the proof of the preceding statements it is enough to consider the case when $H = \{1\}$. So let L be a finitely generated subgroup of G and let $L = L_0 \supset L_1 \supset \cdots \supset L_\alpha \supset \cdots$ be the lower central series of L, that is $L_1 = [L, L]$, $L_{i+1} = [L, L_i]$ for each ordinal number i and $L_\alpha = \bigcap_{\beta<\alpha} L_\beta$ for limit ordinals. Then L is a free group and it is well known that all factor groups $L_\alpha/L_{\alpha+1}$ are torsion free abelian groups [6, p. 232] and $\bigcap_{\alpha\ge 0} L_\alpha = \{1\}$

[6, p. 230]. Hence, in view of Proposition 3.2 (ii) and Lemma 2.4 we obtain $\mathcal{J}(K * L) \subseteq \mathcal{J}(K)(K * L)$. Then the result follows by Lemma 2.3 (i). ∎

Theorem 3.6. *If K is an algebra over the field Q, then for every group G there exists a normal subgroup H such that $H = H'$ and*

$$\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G).$$

Proof. Let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_\alpha \supseteq \cdots$ be the commutator series of G, that is $G_1 = [G, G]$, $G_{\alpha+1} = [G_\alpha, G_\alpha]$ for every ordinal number α and $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$ for limit ordinal numbers α. Then all factor groups $G_\alpha/G_{\alpha+1}$ are abelian and there exists an ordinal number τ such that $G_\tau = G_\alpha$ for all $\alpha \geq \tau$ [6, p. 89]. Now the result follows from Proposition 3.2 (ii) and Lemma 2.4 with $H = G_\tau$, $V_\alpha = G_\alpha$ and $U_\alpha = G_{\alpha+1}$ for all $\alpha \geq 0$. ∎

In particular, if G is a generalized solvable group [5,6] and K is a Q-algebra, then $\mathcal{J}(K * G) \subseteq \mathcal{J}(K)(K * G)$.

Denote by M the set of the locally finite, locally nilpotent, locally free and generalized solvable groups. Then the obtained results can be formulated in the following more general

Theorem 3.7. *Let H be a subgroup of G and let $S(H) = \{U_i, V_i) \mid i \in \Omega\}$ be an H-series of G. If all factors V_i/U_i $(i \in \Omega)$ are M-groups and K is a Q-algebra, then $\mathcal{J}(K * G) \subseteq \mathcal{J}(K * H)(K * G)$.*

4. Radicals of twisted group rings

Let $K_\rho G$ be any twisted group ring of a group G over a ring K with respect to the factor set ρ and let F be the center of K. Then the conditions for associativity of $K_\rho G$ show that $\rho(g, h) \in F$ for all $g, h \in G$. Therefore there is a subring $F_\rho G$ of the ring $K_\rho G$.

In this section we shall consider some properties of the radical $\mathcal{J}(K_\rho G)$. First we shall prove the following

Lemma 4.1. *If K is a central simple F-algebra and I is a non-zero ideal of $K_\rho G$, then $I \cap F_\rho G$ is a non-zero ideal of $F_\rho G$.*

Proof. It suffices to show that $I \cap F_\rho G \neq O$. Let

$$a = \sum_{i=1}^{n} \bar{g}_i \alpha_i \quad (0 \neq \alpha_i \in K)$$

be a non-zero element of I with a minimal support size $\|a\| = n$. By assumption, the principal ideal $K\alpha_1 K$ contains the identity element of K. If

$$\sum_{j=1}^{m} x_j \alpha_1 y_j = 1 \quad (x_j, y_j \in K),$$

then

$$u = \sum_{j=1}^{m} x_j a y_j = \sum_{i=1}^{n} \bar{g}_i \beta_i \quad (\beta_i \in K)$$

is a non-zero element of I, because $\beta_1 = 1$. Thus $v = \alpha u - u\alpha \in I$ and $\|u\| < n$ for all $\alpha \in K$. This means that $v = 0$ and hence $\beta_1, \beta_2, \ldots, \beta_n \in F$, that is $u \in F_\rho G$ and $I \cap F_\rho G \neq O$, as was to be shown. ∎

Lemma 4.2. *Let $K_\rho G$ be a twisted group ring over a central simple F-algebra K of characteristic $p > 0$ and let G be a group with no p-elements. If the factor group G/H is either locally finite, or locally solvable and $\mathcal{J}(K_\rho H) = O$, then $\mathcal{J}(K_\rho G) = O$.*

P r o o f. Assume by way of contradiction that a is a non-zero element of $\mathcal{J}(K_\rho G)$ and let $L_0 = \langle Supp\, a\rangle$ be the supporting subgroup of a. If $L_0 \subseteq H$, then $a \in \mathcal{J}(K_\rho G) \cap K_\rho H \subseteq \mathcal{J}(K_\rho H) = O$, which is impossible. Therefore there exists a subgroup $L = HL_0$ such that $G \supset L \supset H$, $a \in \mathcal{J}(K_\rho L)$ and L/H is a finitely generated factor group. Thus L/H is either finite, or solvable.

If L/H is a finite group of order m, then by Lemma 1.1 we obtain $\mathcal{J}(K_\rho L)^m \subseteq \mathcal{J}(K_\rho H)(K\rho L) = O$, i.e. $\mathcal{J}(K_\rho L)$ is a nilpotent ideal. Now by Lemma 4.1 and Lemma 2.5 we conclude that L has a non-identity p-element, which is a contradiction.

If L/H is a solvable group, then let $H = L_0 \subset L_1 \subset \cdots \subset L_n = L$ be a subnormal series of L with abelian quotients. By induction on $i = 0, 1, \ldots, n$ we shall prove that $\mathcal{J}(K_\rho L_i) = O$. Because $L_0 = H$, the case $i = 0$ is clear. Suppose that $1 \leq i \leq n$ and that $\mathcal{J}(K_\rho L_{i-1}) = O$. If $\mathcal{J}(K_\rho L_i) \neq O$, then it is easy to see as above that there is an intermediate subgroup N such that $L_i \supset N \supset L_{i-1}$, $\mathcal{J}(K_\rho N) \neq O$ and N/L_{i-1} is a finitely generated abelian group. Thus there exists a subgroup N_0 with $N \supset N_0 \supset L_{i-1}$, where N/N_0 is torsion free abelian and N_0/L_{i-1} has a finite order m. Then Lemma 1.1 yields $\mathcal{J}(K_\rho N_0)^m \subseteq \mathcal{J}(K_\rho L_{i-1})(K_\rho N_0) = O$, and hence $\mathcal{J}(K_\rho N_0)$ is nilpotent. Therefore, by Lemma 4.1 and Lemma 2.5, we have $\mathcal{J}(K_\rho N_0) = O$. Furthermore, because N/N_0 is a torsion free abelian group, Proposition 3.2 yields $\mathcal{J}(K_\rho N) \subseteq \mathcal{J}(K_\rho N_0)(K_\rho N) = O$, which is a contradiction. So $\mathcal{J}(K_\rho L_i) = O$ and the induction step is proved. Since $L_n = L$, we obtain $\mathcal{J}(K_\rho L) = O$ and again we receive a contradiction. Therefore $\mathcal{J}(K_\rho G) = O$ and the lemma is proved. ∎

Further by $\mathcal{B}(K)$ we shall denote the Brown-McCoy radical of the ring K, that is $\mathcal{B}(K)$ is the intersection of all maximal ideals in K [1, p. 331].

Theorem 4.3. *Let K be an associative ring of finite characteristic $m > 0$ and let G be a group with no p-elements for every prime divisor p of m. If G has a subnormal series $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ and all factors G_i/G_{i-1} $(i = 1, 2, \ldots, n)$ are either locally finite, or locally solvable, then $\mathcal{J}(K_\rho G) \subseteq \mathcal{B}(K)(K_\rho G)$.*

Proof. Let P be a maximal ideal in K. Then K/P is a simple ring of finite characteristic $p > 0$, where p divides m. It is clear that $P(K_\rho G)$ is an ideal in $K_\rho G$ and $K_\rho G \,/\, P(K_\rho G) = (K/P)_{\bar\rho} G$, where $\bar\rho(g,h) = \rho(g,h) + P$ for all $g, h \in G$. Since G is a group with no p-elements and K/P is a central simple algebra of characteristic p, by Lemma 4.2 with an induction on n we see that $\mathcal{J}((K/P)_{\bar\rho} G) = O$. Therefore, $\mathcal{J}(K_\rho G) \subseteq P(K_\rho G)$ for all maximal ideals P of K. This shows that $\mathcal{J}(K_\rho G) \subseteq \mathcal{B}(K)(K_\rho G)$ and the theorem is proved. ■

Let $K * G$ be again any crossed product and let $Inn(K)$ be the group of the inner automorphisms of K. It is easy to verify that $G_{ker} = \{g \in G \mid g\sigma \in Inn(K)\}$ is a normal subgroup of G and for every element $g \in G_{ker}$ there exists an element $\varepsilon_g \in K^*$ such that $\alpha^{g\sigma} = \varepsilon_g \alpha \varepsilon_g^{-1}$ for all $\alpha \in K$. Setting $\tilde g = \bar g \varepsilon_g$ $(g \in G_{ker})$, we have

$$\alpha \tilde g = \alpha \bar g \varepsilon_g = \bar g \alpha^{g\sigma} \varepsilon_g = \bar g(\varepsilon_g \alpha \varepsilon_g^{-1})\varepsilon_g = \tilde g \alpha$$

for all $\alpha \in K$ and $g \in G_{ker}$. Moreover, $\tilde g \tilde h = \widetilde{gh}\tilde\rho(g,h)$, where $\tilde\rho(g,h) = \varepsilon_{gh}^{-1}\rho(g,h)\varepsilon_h\varepsilon_g$ for all $g, h \in G_{ker}$. Since this is a diagonal change of basis, we conclude that $K * G_{ker} = K_{\tilde\rho} G_{ker}$ is a twisted group ring. Then as an immediate application of the preceding theorem we have the following

Corollary 4.4. *Let $K * G$ be a crossed product over a simple ring K of finite characteristic $p > 0$ and let G_{ker} be a group with no p-elements. If G_{ker} has a subnormal series $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G_{ker}$ such that all factors G_i/G_{i-1} $(i = 1, 2, \ldots, n)$ are either locally finite, or locally solvable, then $\mathcal{J}(K * G) = O$.*

Proof. Assume by way of contradiction that $\mathcal{J}(K * G) \neq O$. Then $I = \mathcal{J}(K * G) \cap K * G_{ker}$ is a non-zero ideal of $K * G_{ker}$ [2] (see also [9, Corollary 12.6]). Moreover, Lemma 2.2 (ii) shows that $I \subseteq \mathcal{J}(K * G_{ker})$ and therefore $\mathcal{J}(K * G_{ker}) \neq O$. But, as was seen in the above notes, $K * G_{ker}$ is a twisted group ring and the preceding theorem shows that $\mathcal{J}(K * G_{ker}) = O$, because $\mathcal{B}(K) = O$. This contradiction proves the correctness of the statement. ■

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Received: 14.10.2002

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