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On 5-Tuples (a, b, c, r, s) with Property M for Even and Corresponding Polynomials

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In this paper, let $d = 2a + b + c$ be even. We find out 5-tuples (a, b, c, r, s) that satisfy property for $d \leq 10$ and obtaine corresponding word and polynomial.

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1. Introduction

Let $F_n =< x_0, x_1, x_2, ..., x_{n-1} >$ be the free group of rank n on free generators $x_0, x_1, x_2, ..., x_{n-1}$. A finite balanced presentation of the group

$$< x_1, x_2, x_3, ..., x_n \,|\, r_1, r_2, r_3, ..., r_n >$$

is said to be a cyclic presentation, if there exists a word w in the free group F_n generated by $x_1, x_2, x_3, ..., x_n$ such that the relators of the presentation are

$$r_k = \theta_n^{(k-1)}(w), \quad k = 1, 2, ..., n,$$

where $\theta_n : F_n \longrightarrow F_n$ denotes the automorphism defined by $\theta_n(x_i) = x_{i+1}$ mod(n), $i = 1, 2, 3, ..., n$. Let us denote this cyclic presentation (and the related group) by the symbol $G_n(w)$, so that

$$G_n(w) =< x_1, x_2, ..., x_n \mid w, \theta_n(w), \theta_n^2(w), ..., \theta_n^{(n-1)}(w))$$

A group is said to be cyclically presented, if it admits a cyclic presentation.

Let $A_n(w) = G_n(w)^{ab}$, where $A_n(w)$ for the derive factor group of $G_n(w)$ the polynomial associated with the cyclically presented group $G = G_n(w)$ is defined to be

$$f(t) = \sum_{i=0}^{n-1} a_i t^i ,$$

where a_i is the exponent sum of x_i in w, $1 \leq i \leq n$.

Let a, b, c, n be integers such that $n > 0$, $a, b, c \geq 0$ and $a + b + c > 0$. Let $\tau(a, b, c)$ be the graph shown in Figure 1. This is an infinite graph with an automorphism θ such that $\theta(u_n) = u_{n+1}$ and $\theta(v_n) = v_{n+1}$. The labeling indicates the number of edges joining a pair of vertices. Thus there are a edges joining u_1 and u_2. We see that $\tau(a, b, c)$ is $d-$ regular, where $d = 2a + b + c$. Let $\tau_n = \tau_n(a, b, c)$ denote the graph obtained from $\tau(a, b, c)$ by identifying all edges and vertices in each orbit of θ^n. Thus τ_n has $2n$ vertices [1].

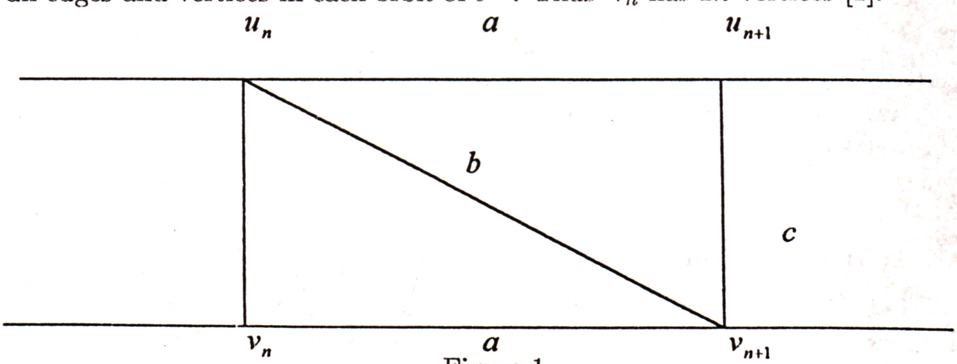

Figure 1

Definition 1.1. If a 6-tuple (a, b, c, r, s, n) corresponds to the Heegaard diagram of a 3-manifold, the 6-tuple (a, b, c, r, s, n) has the property M.

An algorithm for determining which 6-tuples have property M is now described. Put $d = 2a + b + c$ and

$$X = \{-1, -2, -3, \ldots, -d, 1, 2, 3, \ldots, d\}.$$

Let α, β be the permutations of X defined as follows:

$$\alpha(1, d)(2, d-1)\ldots(a, d-a+1)(a+1, -a-c-1)(a+2, -a-c-2)\ldots$$

$$(a+b, -a-c-b)(a+b+1, -a-1)(a+b+2, -a-2)\ldots(a+b+c, -a-c)$$

$$(-1, -d)$$

and

$$\beta(j) = -j + r, \quad \text{if } j + r < 0, \quad \beta(j) = -j + r - d, \quad \text{otherwise.}$$

The cycles of α, which are all 2-cycles, correspond to the end point of line segments in the Heegaard diagram. Each cycle of β corresponds to a pair of endpoints which is identified in forming the surface S.

In general, if the 6-tuple (a, b, c, r, s, n) has the property M, then $\alpha\beta$ is the product of two disjoint cycles of length d, a cycle of $\alpha\beta$ represents the initial point of line segments in an oriented simple closed curve resulting from the identification specified by β. Here r is said to be a rotation factor. Once a rotation factor r is chosen, we are ready to construct Heegaard diagram with cyclic symmetry. Essentially what we have to do is that given a, b and c, to find a pair of rotation factor r and shift factor s. Based on the Neuwirth's algorithm [2], the process of finding such a closed path is neatly described by a product of permutations $\alpha\beta$ on a set

$$X = \{-1, -2, -3, ..., -d, 1, 2, 3, ..., d\}.$$

Indeed, Dunwoody [1] showed that theorem. Let $d = 2a + b + c$ be odd. The 6-tuple (a, b, c, r, s, n) determines a n-genus Heegaard diagram of a closed orientable 3-manifolds, if and only if:

(i) $\alpha\beta$ has two cycles of length d, and

(ii) $ps + q \equiv 0 \pmod{n}$,

where p is the number of arrows pointing down the page minus the number of arrows pointing up, where as q is the number of arrows pointing from left to right minus the number of arrows pointing from right to left in the oriented path determined by $\alpha\beta$. The entries in the first cycle of $\alpha\beta$ contain one vertex from each line segment of the diagram. There exists an integer s such that $ps + q = 0$. The first cycle of $\alpha\beta$ and the value of s can also be used to calculate the word w of the corresponding cyclic presentation the terms in the first cycle of $\alpha\beta$ determines the index of w.

The theorem of Dunwoody is probably true without the restriction that d be odd. That is, the theorem of Dunwoody is true when d is even.

Example . For $a = 1$, $b = 4$, $c = 2$, $d = 2a + b + c = 8$:
$X = \{ -8, -7, -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7, 8\}$
$\alpha = (1, 8)(2, -4)(3, -5)(4, -6)(5, -7)(6, -2)(7, -3)(-8, -1)$

$\beta = (1,-2)(2,-3)(3,-4)(4,-5)(5,-6)(6,-7)(7,-8)(8,-1)$.

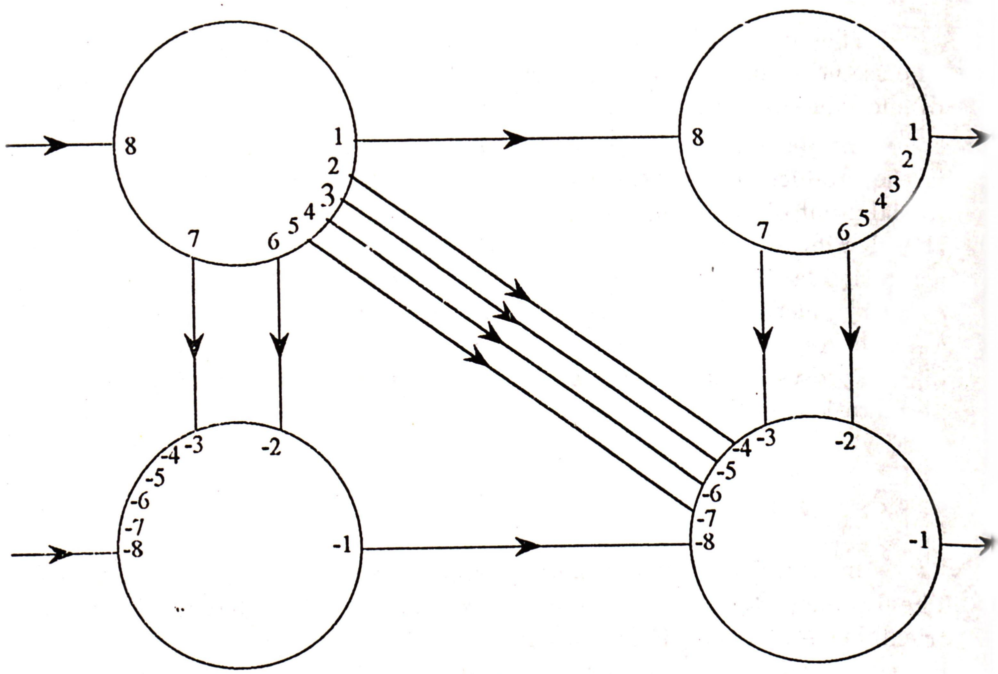

Figure 2

The case of this example is illustrated in Figure 2.

The cycles of α, which are all 2-cycles, correspond to the end point of segments in the Heegaard diagram. Each cycle of β corresponds to a pair of endpoints which is identified in forming the surface S. In this example,

$$\alpha\beta = (1,-1,7,2,3,4,5,6)(-2,-7,-6,-5,-4,-3,-8,8).$$

As in this case, if the 6-tuple (a,b,c,r,s,n) has the property M, then $\alpha\beta$ is the product of two disjoint cycles of length d. For $p=6$ and $q=6, s=-1$:

$0(-1)+1 = x_1^{-1}$
$0(-1)+2 = x_2^1$
$1(-1)+2 = x_1^1$
$2(-1)+3 = x_1^1$
$3(-1)+4 = x_1^1$
$4(-1)+5 = x_1^1$

$$5(-1) + 6 = x_1^1$$
$$6(-1) + 6 = x_0^1$$
$$w = x_0^1 x_1^{-1} x_2^1 x_1^1 x_1^1 x_1^1 x_1^1 x_1^1 \qquad f(t) = 1 + 4t + t^2 \, .$$

The algorithm described above was implemented. All the 4-tuple (a, b, c, r) for which $d = 2a + b + c$ is even and less than 10 and $0 \leq r \leq d$, were successively enumereted. Thus,

(d, a, b, c, r, s)	w	$f(t)$
$(4, 1, 0, 2, 1, -1)$	$x_0 x_1^{-1} x_2 x_1$	$1 + t^2$
$(4, 1, 0, 2, 3, 1)$	$x_1 x_2^{-1} x_1^{-1} x_0$	$-1 - t^2$
$(6, 1, 2, 2, 1, -1)$	$x_1^{-1} x_2 x_1 x_1 x_1 x_0$	$1 + 2t + t^2$
$(8, 1, 0, 6, 5, 1)$	$x_3 x_4^{-1} x_3^{-1} x_2^{-1} x_1^{-1} x_0 x_1 x_2$	$-1 - t^4$
$(8, 1, 0, 6, 3, -1)$	$x_0 x_1^{-1} x_2^{-1} x_3^{-1} x_4 x_3 x_2 x_1$	$1 + t^4$
$(8, 1, 4, 2, 1, -1)$	$x_0 x_1^{-1} x_2 x_1^5$	$1 + 4t + t^2$
$(8, 1, 4, 2, 3, -3)$	$x_4^{-1} x_7^{-1} x_8 x_6 x_4 x_2$	$t^2 + t^6 - t^7 + t^8$
$(8, 3, 0, 2, 3, -3)$	$x_0 x_1^{-1} x_2 x_3^{-1} x_4 x_1 x_2^{-1} x_3$	$1 + t^4$
$(8, 3, 0, 2, 5, 3)$	$x_4^{-1} x_1^{-1} x_2 x_3^{-1} x_0^{-1} x_1 x_2^{-1} x_3$	$-1 - t^4$
$(8, 3, 0, 2, 7, 1)$	$x_0 x_1^{-1} x_0 x_1^{-1} x_2^{-1} x_3^{-1} x_4 x_3^{-1}$	$2 - 2t - t^2 - 2t^3 + t^4$
$(10, 1, 2, 6, 7, 1)$	$x_6^{-1} x_5^{-1} x_4^{-1} x_3^{-1} x_2^{-1} x_1^{-1} x_0^{-1} x_1 x_3 x_5$	$-1 - t^2 - t^4 - t^6$
$(10, 1, 6, 2, 1, -1)$	$x_1^{-1} x_2 x_1^7 x_0$	$1 + 6t + t^2$
$(10, 1, 6, 2, 3, -2)$	$x_1^{-1} x_3^{-1} x_5^{-1} x_6 x_5 x_4 x_3 x_2 x_1 x_0$	$1 + t^2 + t^4 + t^6$
$(10, 1, 6, 2, 5, -1)$	$x_1^{-1} x_2^{-1} x_1^{-1} x_2 x_0^2 x_1 x_2^3$	$2 - t + 3t^2$
$(10, 2, 2, 4, 3, 1)$	$x_6^{-1} x_5^{-1} x_4^{-1} x_3^{-1} x_0^{-1} x_1 x_2 x_3^{-1} x_4 x_5$	$-1 + t^2 - 2t^3 + t^4 - t^6$
$(10, 2, 2, 4, 5, -3)$	$x_1^{-1} x_4^{-1} x_5 x_3 x_1 x_2^{-1} x_5^{-1} x_6 x_3 x_0$	$1 + 2t^3 - t^4 + t^6$
$(10, 2, 4, 2, 7, -2)$	$x_1^{-1} x_2^{-1} x_3 x_4^{-1} x_5^{-1} x_6 x_5 x_3 x_1 x_0$	$1 - t^2 + 2t^3 - t^4 + t^6$
$(10, 3, 2, 2, 1, -1)$	$x_1^{-1} x_2 x_1^{-1} x_2 x_1^3 x_0 x_1^{-1} x_0$	$2 + 2t^2$
$(10, 3, 2, 2, 7, 1)$	$x_2^{-1} x_0 x_2^{-1} x_0^{-1} x_1 x_2^{-1} x_1^{-1} x_0^{-1} x_1$	$-1 + t - 4t^2$
$(10, 3, 2, 2, 3, -2)$	$x_1^{-1} x_2 x_1 x_0 x_1^{-1} x_2 x_0 x_1^{-1} x_2 x_0$	$-3 - 2t + 3t^2$
$(10, 4, 0, 2, 3, -2)$	$x_1^{-1} x_2 x_0 x_1^{-1} x_2 x_0 x_1^{-1} x_2 x_1^{-1} x_0$	$3 - 4t + 3t^2$
$(10, 4, 0, 2, 7, -2)$	$x_2^{-1} x_1 x_0^{-1} x_1 x_2^{-1} x_0^{-1} x_1 x_2^{-1} x_0^{-1} x_1$	$-3 + 4t - 3t^2$

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