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Isogeodesic and Isochebyshevian Nets in a Three-Dimensional Affinely Connected Space without a Torsion

Ivan At. Badev

Presented by P. Kenderov

Isogeodesic and isochebyshevian nets of the second kind into a three-dimensional affinely connected space without a torsion are introduced. Characteristics of the spaces, containing such nets are obtained. We find conforming transformations of a three-dimensional Riemannian space by which an arbitrary net is transformed into isogeodesic or isochebyshevian of the second kind.

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1. Isogeodesic and Isochebyshevian Nets in A_3

Let A_3 be a three-dimensional affinely connected space without a torsion A_3. There is a net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ defined by the independent fields of directions $\underset{\alpha}{v}^i$ $(\alpha = 1, 2, 3)$. The reciprocal co vectors $\overset{\alpha}{v}_i$ of $\underset{\alpha}{v}^i$ are defined by the equations:

$$\overset{\alpha}{v}_i \underset{\alpha}{v}^k = \delta_i^k \iff \underset{\alpha}{v}^i \overset{\beta}{v}_i = \delta_\alpha^\beta .$$

We shall denote the coefficients of the connection of the space A_3 by Γ_{ks}^i. The following derivative formulae hold [3], [4]:

$$(1) \qquad \nabla_k \underset{\alpha}{v}^i = \underset{\alpha}{\overset{\sigma}{T}} k \underset{\sigma}{v}^i, \qquad \nabla_k \overset{\alpha}{v}_i = -\underset{\sigma}{\overset{\alpha}{T}} k \overset{\sigma}{v}_i \quad \text{for any } \alpha, \sigma = 1, 2, 3.$$

1.1. After contracting (1) respectively by $\underset{1}{v}^k$, $\underset{2}{v}^k$ and $\underset{3}{v}^k$, we obtain:

$$\begin{aligned} \underset{1}{v}^k \nabla_k \underset{1}{v}^i &= \overset{1}{\underset{1}{T}}_k \underset{1}{v}^k \underset{1}{v}^i + \overset{2}{\underset{1}{T}}_k \underset{1}{v}^k \underset{2}{v}^i + \overset{3}{\underset{1}{T}}_k \underset{1}{v}^k \underset{3}{v}^i, \\ \underset{2}{v}^k \nabla_k \underset{2}{v}^i &= \overset{1}{\underset{2}{T}}_k \underset{2}{v}^k \underset{1}{v}^i + \overset{2}{\underset{2}{T}}_k \underset{2}{v}^k \underset{2}{v}^i + \overset{3}{\underset{2}{T}}_k \underset{2}{v}^k \underset{3}{v}^i, \\ \underset{3}{v}^k \nabla_k \underset{3}{v}^i &= \overset{1}{\underset{3}{T}}_k \underset{3}{v}^k \underset{1}{v}^i + \overset{2}{\underset{3}{T}}_k \underset{3}{v}^k \underset{2}{v}^i + \overset{3}{\underset{3}{T}}_k \underset{3}{v}^k \underset{3}{v}^i. \end{aligned} \tag{2}$$

Definition 1. The net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in A_3$ will be called an isogeodesic net if:

$$\overset{2}{\underset{1}{T}}_k \underset{1}{v}^k = -\overset{1}{\underset{2}{T}}_k \underset{2}{v}^k, \quad \overset{3}{\underset{2}{T}}_k \underset{2}{v}^k = -\overset{2}{\underset{3}{T}}_k \underset{3}{v}^k, \quad \overset{1}{\underset{3}{T}}_k \underset{3}{v}^k = -\overset{3}{\underset{1}{T}}_k \underset{1}{v}^k. \tag{3}$$

When

$$\overset{2}{\underset{1}{T}}_k \underset{1}{v}^k = -\overset{1}{\underset{2}{T}}_k \underset{2}{v}^k = 0, \quad \overset{3}{\underset{2}{T}}_k \underset{2}{v}^k = -\overset{2}{\underset{3}{T}}_k \underset{3}{v}^k = 0, \quad \overset{1}{\underset{3}{T}}_k \underset{3}{v}^k = -\overset{3}{\underset{1}{T}}_k \underset{1}{v}^k = 0,$$

then the net $(v, v, v) \in A_3$ is a geodesic one [2].

Let a net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ be an isogeodesic one. From (2) and (3) we obtain:

$$\begin{aligned} \overset{1}{v}_i \underset{3}{v}^k \nabla_k \underset{3}{v}^i &= -\overset{3}{v}_i \underset{1}{v}^k \nabla_k \underset{1}{v}^i, \\ \overset{1}{v}_i \underset{2}{v}^k \nabla_k \underset{2}{v}^i &= -\overset{2}{v}_i \underset{1}{v}^k \nabla_k \underset{1}{v}^i, \\ \overset{2}{v}_i \underset{3}{v}^k \nabla_k \underset{3}{v}^i &= -\overset{3}{v}_i \underset{2}{v}^k \nabla_k \underset{2}{v}^i. \end{aligned} \tag{4}$$

Proposition 1. *If a coordinate net* $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in A_3$ *is an isogeodesic one then the coefficients of connection satisfy the equations:*

$$\Gamma^s_{kk} = -\Gamma^k_{ss} \quad \textit{for any} \quad k \neq s, \quad k, s = 1, 2, 3. \tag{5}$$

Proof. From the first equation of (4) we have

$$\overset{1}{v}_i \underset{3}{v}^k \left(\partial_k \underset{3}{v}^i + \Gamma^i_{ks} \underset{3}{v}^s \right) = -\overset{3}{v}_i \underset{1}{v}^k \left(\partial_k \underset{1}{v}^i + \Gamma^i_{ks} \underset{1}{v}^s \right).$$

Since the net $(\underset{1}{v},\underset{2}{v},\underset{3}{v}) \in A_3$ is coordinate, then $\Gamma^3_{11} = -\Gamma^1_{33}$.

The equalities $\Gamma^2_{11} = -\Gamma^1_{22}$, $\Gamma^3_{22} = -\Gamma^2_{33}$ are proved in essentially the same way.

Conversely, if the coefficients of connection of the space A_3, in the parameters of a coordinate net $(\underset{1}{v},\underset{2}{v},\underset{3}{v})$ satisfy (5), then the coordinate net is an isogeodesic one. Really, from (5) taking into account the choice of a coordinate net, we obtain subsequently (4) and (3). From where, it follows that the net $(\underset{1}{v},\underset{2}{v},\underset{3}{v})$ is an isogeodesic one. ∎

1.2. After contracting (1) respectively by $\underset{1}{v}^k$, $\underset{2}{v}^k$ and $\underset{3}{v}^k$, we obtain:

$$\underset{1}{v}^k\nabla_k\overset{1}{v}_i = -\overset{1}{\underset{1}{T}}{}_k\underset{1}{v}^k\overset{1}{v}_i - \overset{1}{\underset{2}{T}}{}_k\underset{1}{v}^k\overset{2}{v}_i - \overset{1}{\underset{3}{T}}{}_k\underset{1}{v}^k\overset{3}{v}_i,$$

$$\underset{2}{v}^k\nabla_k\overset{2}{v}_i = -\overset{2}{\underset{1}{T}}{}_k\underset{2}{v}^k\overset{1}{v}_i - \overset{2}{\underset{2}{T}}{}_k\underset{2}{v}^k\overset{2}{v}_i - \overset{2}{\underset{3}{T}}{}_k\underset{2}{v}^k\overset{3}{v}_i, \tag{6}$$

$$\underset{3}{v}^k\nabla_k\overset{3}{v}_i = -\overset{3}{\underset{1}{T}}{}_k\underset{3}{v}^k\overset{1}{v}_i - \overset{3}{\underset{2}{T}}{}_k\underset{3}{v}^k\overset{2}{v}_i - \overset{3}{\underset{3}{T}}{}_k\underset{3}{v}^k\overset{3}{v}_i.$$

Definition 2. The net $(\underset{1}{v},\underset{2}{v},\underset{3}{v}) \in A_3$ we call an isochebyshevian net of the second kind if:

$$\overset{1}{\underset{3}{T}}{}_k\underset{1}{v}^k = -\overset{3}{\underset{1}{T}}{}_k\underset{3}{v}^k, \quad \overset{1}{\underset{2}{T}}{}_k\underset{1}{v}^k = -\overset{2}{\underset{1}{T}}{}_k\underset{2}{v}^k, \quad \overset{2}{\underset{3}{T}}{}_k\underset{2}{v}^k = -\overset{3}{\underset{2}{T}}{}_k\underset{3}{v}^k. \tag{7}$$

When

$$\overset{1}{\underset{3}{T}}{}_k\underset{1}{v}^k = -\overset{3}{\underset{1}{T}}{}_k\underset{3}{v}^k = 0, \quad \overset{1}{\underset{2}{T}}{}_k\underset{1}{v}^k = -\overset{2}{\underset{1}{T}}{}_k\underset{2}{v}^k = 0, \quad \overset{2}{\underset{3}{T}}{}_k\underset{2}{v}^k = -\overset{3}{\underset{2}{T}}{}_k\underset{3}{v}^k = 0,$$

then the net $(\underset{1}{v},\underset{2}{v},\underset{3}{v}) \in A_3$ is a chebyshevian one of the second kind.

Let the net $(\underset{1}{v},\underset{2}{v},\underset{3}{v}) \in A_3$ be a isochebyshevian of the second kind. From (6) and (7) we have

$$\underset{3}{v}^i\underset{1}{v}^k\nabla_k\overset{1}{v}_i = -\underset{1}{v}^i\underset{3}{v}^k\nabla_k\overset{3}{v}_i,$$

$$\underset{1}{v}^i\underset{2}{v}^k\nabla_k\overset{2}{v}_i = -\underset{2}{v}^i\underset{1}{v}^k\nabla_k\overset{1}{v}_i, \tag{8}$$

$$\underset{2}{v}^i\underset{3}{v}^k\nabla_k\overset{3}{v}_i = -\underset{3}{v}^i\underset{2}{v}^k\nabla_k\overset{2}{v}_i.$$

Proposition 2. *If the coordinate net* $(\underset{1}{v},\underset{2}{v},\underset{3}{v}) \in A_3$ *is isochebyshevian of the second kind then the coefficients of the connection satisfy the equations:*

$$\Gamma^k_{ks} = -\Gamma^s_{sk} \quad \textit{for any } k \neq s,\ k,s = 1,2,3. \tag{9}$$

P r o o f. From the first equation of (8) we obtain :

$$\underset{3}{v}^i \underset{1}{v}^k \left(\partial_k \overset{1}{v}_i - \Gamma^s_{ki} \overset{1}{v}_s\right) = -\underset{1}{v}^i \underset{3}{v}^k \left(\partial_k \overset{3}{v}_i - \Gamma^s_{ki} \overset{3}{v}_s\right),$$

from where, it follows that the coefficients of the connection, in the parameters of the chosen coordinate net, satisfy the equation $\Gamma^1_{13} = -\Gamma^3_{31}$.

Using the same argument we obtain: $\Gamma^2_{21} = -\Gamma^1_{12}$ and $\Gamma^2_{23} = -\Gamma^3_{32}$.

Conversely, if the coefficients of the connection of the space A_3, in the parameters of the of the coordinate net $(\underset{1}{v},\underset{2}{v},\underset{3}{v})$ satisfy the equations (9) then the coordinate net is isochebyshevian one of the second kind. ∎

2. Conforming-isogeodesic and conforming-isochebyshevian nets in a three-dimensional Riemannian space V_3

Let a space A_3 be a three-dimensional Riemannian space V_3 with a metric tensor g_{is}. We consider the conforming transformation:

$$\overset{*}{g}_{is} = e^{2\lambda} g_{is}. \tag{10}$$

An arbitrary net $(\underset{1}{v},\underset{2}{v},\underset{3}{v}) \in V_3(g_{is})$ transforms into $(\underset{1}{\overset{*}{v}},\underset{2}{\overset{*}{v}},\underset{3}{\overset{*}{v}}) \in V_3(\overset{*}{g}_{is})$ by (10). Following Norden [2], for the coefficients of the connections of the spaces $\overset{*}{V}_3$ and V_3, we obtain:

$$\overset{*}{\Gamma}{}^i_{sk} = \Gamma^i_{sk} + \delta^i_s \lambda_k + \delta^i_k \lambda_s - g^{ij} g_{sk} \lambda_j.$$

The vector $\lambda_k = \partial_k \lambda = \dfrac{\partial \lambda}{\partial u^k}$ is called the vector of conform transformation. It is known that the conform transformation of a Riemannian space is characterized by the condition $\lambda_k =$ grad.

Let the derivative formulae in the space $\overset{*}{V}_3$ are:

$$\overset{*}{\nabla}_k \underset{\alpha}{v}^i = \underset{\alpha}{\overset{\sigma}{P}}{}_k \underset{\sigma}{v}^i \quad \text{for any } \alpha, \sigma = 1,2,3. \tag{11}$$

2.1. Let the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_3$ be transformed into an isogeodesic one $(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \underset{3}{\overset{*}{v}}) \in \overset{*}{V}_3$ by the conforming transformation (10).

Definition 3. A net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_3$ allowing conforming transformation into an isogeodesic one $(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \underset{3}{\overset{*}{v}}) \in \overset{*}{V}_3$, we shall call a conforming-isogeodesic net.

Theorem 1. *A net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_3$ is conforming-isogeodesic if and only if*

$$\lambda_k = \tilde{\mathbf{Z}}_k^{\gamma} \mathbf{a}_\gamma = \text{grad}\,.$$

Proof. Following [3] and [1] we have $\overset{\beta}{\underset{\alpha}{P}}_k = \overset{\beta}{\underset{\alpha}{T}}_k + \lambda_s \left(v^s \overset{\beta}{\underset{\alpha}{v}}_k - \overset{\beta}{v}_i g^{is} \underset{\alpha}{v}_k \right)$ from where it follows $\overset{\beta}{\underset{\alpha}{P}}_k v^k = \overset{\beta}{\underset{\alpha}{T}}_k \underset{\alpha}{v}^k - \lambda_s \overset{\beta}{v}_i g^{is}$.

Since the net $(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \underset{3}{\overset{*}{v}})$ is isogeodesic then from (3) we can write:

$$\overset{\beta}{\underset{\alpha}{P}}_k \underset{\alpha}{v}^k = -\overset{\alpha}{\underset{\beta}{P}}_k \underset{\beta}{v}^k \quad \text{for any } \alpha \neq \beta.$$

Hence for the vector of the conforming transformation we have:

$$\lambda_s \left(\overset{\alpha}{v}_i g^{is} + \overset{\beta}{v}_i g^{is} \right) = \overset{\beta}{\underset{\alpha}{T}}_k \underset{\alpha}{v}^k + \overset{\alpha}{\underset{\beta}{T}}_k \underset{\beta}{v}^k \quad \text{for any } \alpha \neq \beta,\ \alpha, \beta = 1, 2, 3. \tag{12}$$

Let us introduce the notations:

$$\mathbf{a}_\gamma = \overset{\beta}{\underset{\alpha}{T}}_k \underset{\alpha}{v}^k + \overset{\alpha}{\underset{\beta}{T}}_k \underset{\beta}{v}^k \tag{13}$$

and

$$\mathbf{Z}_\gamma^s = \overset{\alpha}{v}_i g^{is} + \overset{\beta}{v}_i g^{is}, \tag{14}$$

where $\alpha \neq \beta \neq \gamma$; $(\alpha, \beta, \gamma) = (1, 2, 3),\ (2, 3, 1),\ (3, 1, 2)$.

We will prove that λ_s is the solution of the equation:

$$\mathbf{Z}_\gamma^s \lambda_s = \mathbf{a}_\gamma. \tag{15}$$

Since the matrix $(\mathbf{Z}_\gamma^s)$ is a nonsingular one, it has the converse one $(\tilde{\mathbf{Z}}_k^\gamma)$. Then from (15) we find:

$$\lambda_k = \tilde{\mathbf{Z}}_k^\gamma \mathbf{a}_\gamma. \tag{16}$$

2.2. Let us find the conforming transformation such that an arbitrary net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_3(g_{is})$ is transformed into an isochebyshevian net of the second kind $(\overset{*}{\underset{1}{v}}, \overset{*}{\underset{2}{v}}, \overset{*}{\underset{3}{v}}) \in \overset{*}{V}_3(\overset{*}{g}_{is})$.

Definition 4. A net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_3$, allowing conforming transformation into an isochebyshevian one of the second kind $(\overset{*}{\underset{1}{v}}, \overset{*}{\underset{2}{v}}, \overset{*}{\underset{3}{v}}) \in \overset{*}{V}_3$, will be called a conforming-isochebyshevian net of the second kind.

Theorem 2. *A net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_3$ is a conforming-isochebyshevian one of the second kind if and only if $\lambda_k = \tilde{m}_k^{\gamma} b_\gamma =$ grad .*

Proof. Taking into account the chosen conforming transformation, for the coefficients of the derivative formulae in the space $\overset{*}{V}_3$ and V_3 we obtain [2], [3], [1]:

$$(17)\quad \underset{\alpha}{\overset{(\beta)}{P}}{}_k \underset{(\beta)}{v}{}^k = \underset{\alpha}{\overset{(\beta)}{T}}{}_k \underset{(\beta)}{v}{}^k + \lambda_s \left(\underset{\alpha}{v}{}^s - \overset{(\beta)}{v}{}^i g^{is} \cos \underset{\alpha(\beta)}{\omega} \right) \quad \text{for any } \alpha \neq \beta;\ \alpha, \beta = 1, 2, 3,$$

where $\underset{\alpha\beta}{\omega}$ is the angle between the fields $\underset{\alpha}{v}{}^i$ and $\underset{\beta}{v}{}^i$.

(The branched indexes are not to be summed.)

From (17), in accordance with (7), we find

$$\lambda_s \left[\left(\overset{(\alpha)}{v}{}_i g^{is} \cos \underset{(\alpha)\beta}{\omega} + \overset{(\beta)}{v}{}_i g^{is} \cos \underset{\alpha(\beta)}{\omega} \right) - \left(\underset{\alpha}{v}{}^s + \underset{\beta}{v}{}^s \right) \right]$$

$$(18)\quad = \underset{\beta}{\overset{(\alpha)}{T}}{}_k \underset{(\alpha)}{v}{}^k + \underset{\alpha}{\overset{(\beta)}{T}}{}_k \underset{(\beta)}{v}{}^k.$$

Denote

$$b_\gamma = \underset{\beta}{\overset{(\alpha)}{T}}{}_k \underset{(\alpha)}{v}{}^k + \underset{\alpha}{\overset{(\beta)}{T}}{}_k \underset{(\beta)}{v}{}^k$$

and

$$m_\gamma^s = \overset{(\alpha)}{v}{}_i g^{is} \cos \underset{(\alpha)\beta}{\omega} + \overset{(\beta)}{v}{}_i g^{is} \cos \underset{\alpha(\beta)}{\omega} - \left(\underset{\alpha}{v}{}^s + \underset{\beta}{v}{}^s \right),$$

where $\alpha \neq \beta \neq \gamma$; $(\alpha, \beta, \gamma) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$,

$$(19)\quad m_\gamma^s \lambda_s = b_\gamma.$$

Since the matrix (m_γ^s) is a nonsingular one then it has the converse one $(\tilde{m}_\gamma^s)$. Now from (19) it follows that $\lambda_k = \tilde{m}_k^\gamma b_\gamma$, which means that λ_s is the solution of the (19).

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Technical College "John Atanasov"
71 A, Br. Buckston str.
4000, Plovdiv, tel: 60 81 41
BULGARIA

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Corr. address:

Trakia, bl. 61, entr. B, apt. 25
4023, Plovdiv, tel: 82 34 75
BULGARIA