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Some Remarks on Weak β-Openness

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1. Introduction

The concept of β-openness was introduced by M. E. Abd El. Monsef et al. in [1]. These sets are also called semipreopen by D. Andrijevic [3]. Recently M. Caldas and G. Navalagi [5] introduced a new class of functions called weakly β-open functions which contain the class of β-open functions. They remarked that weakly β-open functions are not always β-open. The present paper has as purpose obtain a sufficient condition for a weakly β-open function to be β-open, establish relationships between this function and other generalized forms of openness and also show that the inverse image surjective of every compact set from the codomain are quasi H-closed.

Throughout the present paper, (X, τ) and (Y, σ) (or X and Y) denote topological spaces in which no separation axioms are assume unless explicitely stated. The subset A of the topological space (X, τ) is called β-open [1] or semipreopen [3] if $A \subset Cl(Int(Cl(A)))$, where $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by $Cl_\beta(A)$. The β-interior of A is the union of all β-open sets contained in A and is denoted by $Int_\beta(A)$. For other definitions and notations of this paper, we refer to [2, 4, 6, 7, 8, 9, 10, 11, 13].

2. Weakly β-open functions

Definition 1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly β-open [5] if for each $x \in X$ and each open set U of X containing x, there exists a β-open set V containing $f(x)$ such that $V \subset f(Cl(U))$. We denote a weakly β-open function by $w.\beta.o.$

The implications between weakly β-open functions and other types of open functions are given by the following diagram (The converse of these statements are not necessarily true).

$$\begin{array}{ccccccc} & & \text{preopen} & & & & \\ & & \uparrow & \searrow & & & \\ \text{open functions} & \rightarrow & \alpha\text{-open} & \rightarrow & \beta\text{-open} & \rightarrow & \text{weakly } \beta\text{-open} \\ & & \downarrow & \nearrow & & & \\ & & \text{semiopen} & & & & \end{array}$$

Example 2.1. (i) Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{c\}\}$ and $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Then f is $w.\beta.o.$ but it is not weakly open since $f(\{a\}) \not\subset Int(f(Cl(\{a\})))$.

(ii) Let $X = \{a, b\}$, $\tau = \{\emptyset, X, \{b\}\}$, $Y = \{x, y\}$ and $\sigma = \{\emptyset, Y, \{x\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is clearly $w.\beta.o.$ but it is not a β-open function since $f(b)$ is not a β-open set in Y.

Example 2.2. (i) An injective function from a discrete space into an indiscrete space is β-open, but is not α-open.

(ii) Let $X = \{x, y, z\}$ and $\tau = \{\emptyset, \{x\}, \{x, y\}, X\}$. Then a function $f : (X, \tau) \rightarrow (X, \tau)$ which is defined by $f(x) = x$, $f(y) = z$ and $f(z) = y$ is α-open but is not open .

Example 2.3. It is known in [2]:

(i) Let $X = Y = \{x, y, z\}$ with $\tau = \{\emptyset, \{x\}, \{x, y\}, X\}$ and σ be an indiscrete topology. Then the identity function $I : (X, \tau) \rightarrow (Y, \sigma)$ is β-open but not semi-open.

(ii) Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $Y = \{a, b, c, d\}$ with topology $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as $f(a) = b$, $f(b) = d$ and $f(c) = a$, it is clear that f is β-open but not preopen.

Theorem 2.4. *For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following are equivalent ([5 ,Theorem 2.3 and 2.4]):*

1) f is $w.\beta.o.$

2) $f(U) \subset Int_\beta(f(Cl(U)))$ for every $U \in \tau$.

3) $f(Int_\theta(A)) \subset Int_\beta(f(A))$ for every $A \subset X$.

4) $Int_\theta(f^{-1}(B)) \subset f^{-1}(Int_\beta(B))$ for every $B \subset Y$.

5) $f^{-1}(Cl_\beta(B)) \subset Cl_\theta(f^{-1}(B))$ for every $B \subset Y$.

6) $f(Int(F)) \subset Int_\beta(f(F))$ for every $F^c \in \tau$.

7) $f(Int(Cl(U))) \subset Int_\beta(f(Cl(U)))$ *for every* $U \in \tau$.
8) $f(U) \subset Int_\beta(f(Cl(U)))$ *for every preopen subset* U *of* X.
9) $f(U) \subset Int_\beta(f(Cl(U)))$ *for every* $U \in \tau^\alpha$.

Theorem 2.5. *For a function* $f : (X,\tau) \to (Y,\sigma)$, *the following are equivalent:*
1) f *is w.β.o.*
2) $f(U) \subset Cl(Int(Cl(f(Cl(U)))))$ *for every* $U \in \tau$.
3) $f(Int_\theta(A)) \subset Cl(Int(Cl(f(A))))$ *for every* $A \subset X$.
4) $Int_\theta(f^{-1}(B)) \subset f^{-1}(Cl(Int(Cl(B))))$ *for every* $B \subset Y$.
5) $f^{-1}(Int(Cl(Int(B)))) \subset Cl_\theta(f^{-1}(B))$ *for every* $B \subset Y$.
6) $f(Int(F)) \subset Cl(Int(Cl(f(F))))$ *for every* $F^c \in \tau$.
7) $f(Int(Cl(U))) \subset Cl(Int(Cl(f(Cl(U)))))$ *for every* $U \in \tau$.

Proof. This follows from Theorem 2.4 and that

$$Cl_\beta(A) = A \cup Int(Cl(Int(A)))\,, Int_\beta(A) = A \cap Cl(Int(Cl(A))).$$

Theorem 2.6. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a bijective function. Then the following statements are equivalent:*
1) f *is w.β.o.*
2) $Cl_\beta(f(U)) \subset f(Cl(U))$ *for each* $U \in \tau$.
3) $Cl_\beta(f(Int(F)) \subset f(F)$ *for each* F *closed of* X.

Proof. (1) $\to$ (3) : Let F be a closed set in X. Then we have $f(F^c) = [f(F)]^c \subset Int_\beta(f(Cl(F^c)))$ and so $[f(F)]^c \subset [Cl_\beta(f(Int(F)))]^c$. Hence $f(F) \supset Cl_\beta(f(Int(F)))$.
(3) $\to$ (2) : Let $U \in \tau$. Since $Cl(U)$ is a closed set and $U \subset Int(Cl(U))$ by (3) we have $Cl_\beta(f(U)) \subset Cl_\beta(f(Int(Cl(U)))) \subset f(Cl(U))$.
(2) $\to$ (3) : Similar to (3) $\to$ (2). And (3) $\to$ (1) is clear.

Theorem 2.7. *Let* $g : (Y,\sigma) \to (Z,\gamma)$ *be an open continuous injective. Then a function* $f : (X,\tau) \to (Y,\sigma)$ *is w.β.o. if and only if* $g\circ f : (X,\tau) \to (Z,\gamma)$ *is w.β.o.*

Proof. *Necessity.* Suppose that f is *w.β.o.* Let U be any open set (X,τ). By Theorem 2.4 $f(U) \subset Cl(Int(Cl(f(Cl(U)))))$. Since g is open and continuous, we have $g(Cl(Int(Cl(A)))) \subset Cl(Int(Cl(g(A))))$ for every subset A of Y. Therefore, we obtain $(g \circ f)(U) \subset Cl(Int(Cl(g \circ f)(Cl(U))))$. It follows from Theorem 2.4 that $g \circ f$ is *w.β.o.*
Sufficiency. Suppose that $g \circ f$ is *w.β.o.* Let U be any open set of (X,τ). By Theorem 2.4, $(g \circ f)(U) \subset Cl(Int(Cl(g \circ f)(Cl(U))))$. Since g is open and continuous, we have $g^{-1}(Cl(Int(Cl(A)))) \subset Cl(Int(Cl(g^{-1}(A))))$ for every subset

A of Z. Moreover, singe g is injective we obtain $f(U) \subset Cl(Int(Cl(f(Cl(U)))))$. It follows from Theorem 2.4, that f is $w.\beta.o.$

Theorem 2.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function.*
1) If f has the property that for each regular closed set A, $f(A)$ is a β-open of Y, then f is $w.\beta.o.$
2) If f is $w.\beta.o.$, then $f(B)$ is a β-open set for each clopen set B.

Remark 2.9. If X is an extremally disconnected space, then both converses of Theorem 2.8 are hold since the regular closed sets are precisely the clopen sets.

Definition 2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to satisfy the weakly β-open interiority condition if $Int_\beta(f(Cl(U))) \subset f(U)$ for every $U \in \tau$.

Example 2.10. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a\}, [a, b\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function and let $U = \{a\}$. Since $Int_\beta(f(Cl(U))) = Int_\beta(f(X)) = X \not\subset f(U) = \{a\}$, f does not satisfy the weakly β-open interiority condition. However, f is clearly β-open.

Theorem 2.11. *If $f : (X, \tau) \to (Y, \sigma)$ is $w.\beta.o.$ and satisfies the weakly β-open interiority condition, then f is β-open.*

Example 2.10, shows that neither of these interiority conditions yields a decomposition of β-openness.

Theorem 2.12. *A function $f : (X, \tau) \to (Y, \sigma)$ is $w.\beta.o.$ with a submaximal and extremally disconnected codomain if and only if f is weakly open.*

Proof. It is an immediate consequence of that every β-open subset of X is open if and only if X is submaximal and extremally disconnected [7].

Lemma 2.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective $w.\beta.o.$ function. If U is a clopen set of (X, τ), then $f(U)$ is β-clopen in (Y, σ).*

Theorem 2.14. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective $w.\beta.o.$ function of a space (X, τ) onto a β-connected space (Y, σ), then (X, τ) is connected.*

Proof. Let us assume that X is not connected. Then there exist nonempty open sets U_1 and U_2 such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = X$. Therefore U_1 and U_2 are clopen in (X, τ) and by Lemma 2.13 $f(U_i)$ is a β-open subset for $i = 1, 2$. Moreover, we have $f(U_1) \cap f(U_2) = \emptyset$ and $f(U_1) \cup f(U_2) = Y$. Since f is bijective , $f(U_i)$ is nonempty for $i = 1, 2$. This indicates that (Y, σ) is not β-connected. This is a contradiction.

Recall that a subset of a topological space is called closure compact (or quasi H-closed [14]) if each open cover of the set contain a finite subcollection whose closures cover the set.

Theorem 2.15. *Let $f : (X, \tau) \to (Y, \sigma)$ be surjective w.β.o. and let K be a compact set of Y. Then $f^{-1}(K)$ is a closure compact subset of X.*

Proof. Let $\Lambda = \{V_\alpha / \alpha \in I\}$, I being the index set be a open cover of $f^{-1}(K)$ and set $T = \{U \in \Lambda / U \cap f^{-1}(K) \neq \emptyset\}$. Then T is an open cover of $f^{-1}(K)$. For each $y \in K$, $f^{-1}(y) \in U_y$ for some $U_y \in T$. By weakly β-openness of f, there exists a β-open subset W_y containing y such that $W_y \subset f(Cl(U_y))$. The collection $\{W_y / y \in K\}$ is a β-open cover of K and so there is a finite subcover $\{W_y / y \in K_0\}$ where K_0 is a finite subset of K. Clearly $\{Cl(U_y) / y \in K_0\}$ covers $f^{-1}(K)$, or $f^{-1}(K)$ is a closure compact subset in X.

References

[1] M.E. Abd. El-Monsef, S.N. El-Deeb and R.A. Mahmoud, β-open and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.*, **12** (1983), 70-90.

[2] D. Andrijevic, Some properties of the topology of α-sets, *Mat. Vesnik*, **36** (1984), 1-10.

[3] D. Andrijevic, Semipreopen sets, *Mat. Vesnik*, **38** (1986), 24-32.

[4] N. Biswas, On some mappings in topological spaces, *Bull. Calcutta Math. Soc.*, **61** (1969), 127-135.

[5] M. Caldas and G. Navalagi, Weakly β-open functions between topological spaces. (In preparation).

[6] M. Caldas and S. Jafari, Some properties of contra β- continuous, *Mem. Fac. Sci. Kochi Univ. Japan*, **22** (2001), 19-28.

[7] M. Ganster and D. Andrijevic, On some questions concerning semipreopen sets, *J. Inst. Math. Comput. Sci. Math. Ser.* **1** (1988), 19-29.

[8] N. Levine, Semiopen sets and semicontinuity in topological space, *Amer. Math. Monthly*, **70** (1963), 36-41.

[9] A.S. Mashhour, M.E. Abd EL-Monsef and S.N. EL-Deeb, Precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egipt*, **53** (1982), 47-53.

[10] A.S. Mashhour, I.A. Hasanein and S.N. EL-Deeb, α-Continuous and α-open mappings, *Acta Math. Hung.*, **41** (1983), 213-218.

[11] O. Najästad, On some classes of nearly open sets, *Pacific J. Math.*, **15** (1965), 961-970.

[12] V. Popa and T. Noiri, Weakly β-continuous functions, *An. Univ. Timisoara Ser. Mat. Inform.*, **32** (1994), 83-92.

[13] D.A. Rose, Weak openness and almost openness, *Internat. J. Math. & Math. Sci.*, **7** (1984), 35-40.

[14] N.V. Velicko, H-Closed topological spaces, *Amer. Math. Soc. Transl.*, **78** (1968), 103-118.

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