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Subscalarity of a Class of Triangular Operator Matricies [1]

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Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space. An operator T in $\mathcal{L}(\mathcal{H})$ is called totally P-posinormal (see [2]) if and only if there is a polynomial $P(z)$ with zero constant term such that $\|(P(T-z))^*h\| \leq M(z)\|(T-z)h\|$ for each $h \in \mathcal{H}$, where $M(z)$ is bounded on compact subsets of $\mathbf{C}$. In this paper we prove that every finite triangular operator matrix $T \in \mathcal{L}(\oplus_1^n \mathcal{H})$ with totally P-posinormal operators on main diagonal is subscalar, i.e. it is a restriction of a generalized scalar operator to an invariant subspace.

1. Introduction

Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all linear continuous operators acting on $\mathcal{H}$. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be *generalized scalar of order m* if it possesses a spectral distribution of order m, i.e. if there is a unital, continuous morphism of topological algebras

$$\Phi : C_0^m \to \mathcal{L}(\mathcal{H}),$$

such that $\Phi(z) = S$, where z is the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ is the set of all compactly supported functions on $\mathbf{C}$, with continuous derivatives of order m $(0 \leq m \leq \infty)$. The operator S is said to be *subscalar* if it is similar to the restriction of a generalized scalar operator to a closed invariant subspace. For further information about these two classes of operators see [3].

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be

- if and only if $TT^* \leq T^*T$ or equivalently, if $\|T^*h\| \leq \|Th\|$ for each $h \in \mathcal{H}$.

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In [5], M.Putinar has proved that hyponormal operators are subscalar of order 2.

It is well known that if T is hyponormal operator, then $\mathrm{Im}(T) \subset \mathrm{Im}(T^*)$. That leads to the following two generalizations. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be:

- if and only if $\mathrm{Im}(T) \subset \mathrm{Im}T^*$, or equivalently $TT^* \leq \lambda^2 T^*T$, for some $\lambda \geq 0$ ([6]);

- $(P-)$ if and only if there is a polynomial $P(z)$ with zero constant term such that

$$\mathrm{Im}(P(T)) \subset \mathrm{Im}(T^*)\,, \text{ or equivalently } \|(P(T))^* h\| \leq \lambda^2 \|Th\|$$

for each $h \in \mathcal{H}$ (see [2]).

Note that every posinormal operator is P-posinormal with $P(z) = z$.

The class of hyponormal operators is invariant by translation, i.e. if the operator T is hyponormal, then $T - z$ is also hyponormal. Since the class of P-posinormal operators does not preserve this important property further requirements are imposed. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be:

- $P-$ if and only if there is a polynomial $P(z)$ with zero constant term such that $\|(P(T - zI))^* h\| \leq M(z)\|(T - zI)h\|$ for each $h \in \mathcal{H}$, where $M(z)$ is bounded on the compacts of $\mathbf{C}$ (see Definition 1 below).

In [2], a functional model for totally P-posinormal operators is obtained and in [4] is proved that they are subscalar.

In [1], E. Ko has considered operator matrix $T \in \mathcal{L}(\oplus_1^n \mathcal{H})$ of the form

$$(1) \qquad T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & T_{22} & \cdots & T_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & T_{nn} \end{pmatrix},$$

where T_{ii} , $i = 1, 2, \ldots n$, are hyponormal operators and proved that T is subscalar of order $2n$ ([1], Theorem 4.5).

The object of this paper is to prove that if T_{ii}, $i = 1, 2, \ldots n$, are totally P-posinormal, then the operator T from (1) is subscalar.

2. Preliminaries

Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space and let $U \subset \mathbf{C}$ be a bounded open set. Troughout this paper we will need the following functional spaces:

- $C^m(U), (0 \leq m \leq \infty)$ - the space of all m-times continuously differentable on U functions; $C^m(U,\mathcal{H}) = C^m(U)\hat{\otimes}\mathcal{H}$ - the space of all $\mathcal{H}$-valued C^m- functions; $C_0^m(U)$ (resp. $C_0^m(U,\mathcal{H})$)-the space of all compact supported functions in C^m (resp. $C^m(U,\mathcal{H})$). We consider these spaces with the usual norms.

- $L_p(U,\mathcal{H})$, $(1 \leq p \leq \infty)$ - the space of all H-valued functions in L_p with the usual norm $\| * \|_{p,U}$.

- $A_2(U,\mathcal{H})$ - the space of all analytic functions from $L_2(U,\mathcal{H})$.

- $W_2^m(U,\mathcal{H}) = \{f \in L_2(U,\mathcal{H}) : \overline{\partial}^k f \in L_2(U,\mathcal{H}), k = 1,2,\ldots,m\}$, $m \in \mathbf{N}$- the so called $\mathcal{H}$-valued Sobolev-type space. This space is Hilbert with respect to the norm $\|f\|_{W_{2,U}^m} = \sum_{k=0}^m \|\overline{\partial}^k f\|_{2,U}$.

We recall that an operator $T \in \mathcal{L}(\mathcal{H})$ has SVEP (single-valued extension property) if for any open set $U \subset \mathbf{C}$ the only analytic solution of the equation $(T-z)f(z) = 0$, $z \in U$, is $f(z) \equiv 0$ in U (see [3]). If, in addition, the range of $T-z$ is closed, then T has property (β) (see [3]).

Definition 1. *([2]). The operator $T \in \mathcal{L}(\mathcal{H})$ is said to be totally P-posinormal if and only if*

$$\|\overline{P}(T^* - \overline{z})h\| \leq M(z)\|(T-z)h\| \text{ for each } h \in \mathcal{H},$$

where $P(z) = z^n + \sum_{k=1}^{n-1} a_k z^{n-k}$, $\overline{P}(\overline{z}) = \overline{z}^n + \sum_{k=1}^{n-1} \overline{a}_k \overline{z}^{n-k}$, and $M(z)$ is bounded on compact subsets of $\mathbf{C}$.

We will need the following lemma.

Lemma 2. *([4], Corollary 1) Let D be an open disk, let T be a totally P-posinormal operator and let S be the orthogonal projection of $L_2(D,\mathcal{H})$ onto $A_2(D,\mathcal{H})$. Then for $f \in W_2^m(D,\mathcal{H})$*

$$\|(I-S)f\|_{2,D} \leq C_D \sum_{k=0}^{n} \|(T-z)\overline{\partial}^{n+k} f\|_{2,D},$$

where C_D is a constant depending on the disk D only.

Let us denote $W_T^{2n}(D,\mathcal{H}) = W_2^{2n}(D,\mathcal{H})/\overline{(T-z)W_2^{2n}(D,\mathcal{H})}$, where the operator T is totally P-posinormal and let $V : \mathcal{H} \to W_T^{2n}(D,\mathcal{H})$ be the operator

$$Vh = 1 \otimes h + (T-z)W_2^{2n}(D,\mathcal{H}), \; h \in \mathcal{H}.$$

If M_z is the multiplication operator with z on $W_2^{2n}(D,\mathcal{H})$ and $\widetilde{M}_z$ is the induced operator by M_z in $W_T^{2n}(D,\mathcal{H})$, then $VT = \widetilde{M}_z V$.

The following lemma is true.

Lemma 3. *([4], Lemma 3) If D is an open disk containing $\sigma(T)$ (the spectrum of T), then the operator V is injective with closed range.*

To prove our main result we will also need

Lemma 4. *([4], Proposition 1) Let T be a totally P-posinormal operator. Then T has property (β).*

3. Main result

Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space and $T \in \mathcal{L}(\oplus_1^n \mathcal{H})$ is a finite triangular operator matrix of the form (1), where T_{ii} , $i = 1, 2, ..., n$, are totally P-posinormal (see definition 1) with corresponding polynomials $P_i(z)$, of degree $deg P_i = m_i$. Let $s = \sum_{i=1}^n m_i$.

Now, let D be an open disk. Denote

$$W_T(D,\mathcal{H}) = \oplus_1^n W_2^{2s}(D,\mathcal{H}) / \overline{(T-z)(\oplus_1^n W_2^{2s}(D,\mathcal{H}))}$$

and let $V : \oplus_1^n \mathcal{H} \to W_T(D,\mathcal{H})$ be the operator determined by the equality

$$Vh = 1 \otimes h + (T-z)(\oplus_1^n W_2^{2s}(D,\mathcal{H})), \ \ h = (h_1, h_2, ..., h_n) \in \oplus_1^n \mathcal{H}. \tag{2}$$

If M_z is the multiplication operator with z on $\oplus_1^n W_2^{2s}(D,\mathcal{H})$ and $\widetilde{M}_z$ is the operator induced by M_z in $W_T(D,\mathcal{H})$, then it is clear that

$$VT = \widetilde{M}_z V. \tag{3}$$

Lemma 5. *Let D be an open disk containing $\sigma(T)$ (the spectrum of T). Then the operator V defined by (2) is injective with closed range.*

P r o o f. Let $\{h^k\}_1^\infty \subset \oplus_1^n \mathcal{H}$, $h^k = (h_1^k, ..., h_n^k)$ and $\{f^k\}_1^\infty \subset \oplus W_2^{2s}(D,\mathcal{H})$, $f^k = (f_1^k, ..., f_n^k)$, be sequences such that

$$\lim_{k\to\infty} \|(T-z)f^k + 1 \otimes h^k\|_{\oplus_1^n W_2^{2s}} = 0. \tag{4}$$

The lemma will be proved if we show that $h^k \to 0$, as $k \to \infty$. Note that condition (4) might be written in the form

$$\begin{cases} \lim_{k\to\infty} \|(T_{11}-z)f_1^k + T_{12}f_2^k + \cdots + T_{1n}f_n^k + 1 \otimes h_1^k\|_{W_2^{2s}} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lim_{k\to\infty} \|(T_{jj}-z)f_j^k + T_{jj+1}f_{j+1}^k + \cdots + T_{jn}f_n^k + 1 \otimes h_j^k\|_{W_2^{2s}} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lim_{k\to\infty} \|(T_{nn}-z)f_n^k + 1 \otimes h_n^k\|_{W_2^{2s}} = 0. \end{cases} \tag{5}$$

We will prove the lemma inductively by a procedure similar to that employed in [1]. We will show that for every $t = 1, 2, ..., n$

$$(6)\quad \begin{cases} \lim_{k\to\infty} \|(T_{11} - z)f_1^k + T_{12}f_2^k + \cdots + T_{1t}f_t^k + 1\otimes h_1^k\|_{W_2^{2s_t}} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lim_{k\to\infty} \|(T_{jj} - z)f_j^k + T_{jj+1}f_{j+1}^k + \cdots + T_{jt}f_t^k + 1\otimes h_j^k\|_{W_2^{2s_t}} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lim_{k\to\infty} \|(T_{tt} - z)f_t^k + 1\otimes h_t^k\|_{W_2^{2s_t}} = 0, \end{cases}$$

where $s_t = s - (m_{t+1} + \cdots + m_n)$. It is clear that in the case $t = n$ (6) is exactly (5).

Suppose that (6) is proved for some $t = 2, 3, ..., n$. We will prove that

$$(7)\quad \begin{cases} \lim_{k\to\infty} \|(T_{11}-z)f_1^k+T_{12}f_2^k+\cdots+T_{1t-1}f_{t-1}^k+1\otimes h_1^k\|_{W_2^{2s_{t-1}}} =0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lim_{k\to\infty} \|(T_{jj}-z)f_j^k+T_{jj+1}f_{j+1}^k+\cdots+T_{jt-1}f_{t-1}^k+1\otimes h_j^k\|_{W_2^{2s_{t-1}}} =0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lim_{k\to\infty} \|(T_{t-1t-1}-z)f_{t-1}^k+1\otimes h_{t-1}^k\|_{W_2^{2s_{t-1}}} =0. \end{cases}$$

By Lemma 3, from (6) we have $h_t^k \to 0$, as $k \to \infty$. Let S be the projection of $\oplus_1^{2s_t-1} L_2(D, \mathcal{H})$ onto $\oplus_1^{2s_t-1} A_2(D, \mathcal{H})$. Then applying Lemma 2 for $T = T_{tt}$ we get

$$\|(I - S)(f_t^k, \overline{\partial} f_t^k, \ldots, \overline{\partial}^{2s_t-1} f_t^k)\|_{2,D}$$

$$\leq C \sum_{k=0}^{m_t} \|(T_{tt} - z)\overline{\partial}^{k+m_t}(f_t^k, \overline{\partial} f_t^k, \ldots, \overline{\partial}^{2s_t-1} f_t^k)\|_{2,D}.$$

Thus, we have

$$(8)\qquad \lim_{k\to\infty} \|(I - S)\overline{\partial}^i f_t^k\|_{2,D} = 0,\ i = 0, 1, ..., 2s_{t-1}.$$

So, for each $i = 0, 1, ..., 2s_{t-1}$

$$\lim_{k\to\infty} \|(T_{tt} - z)(I - S)\overline{\partial}^i f_t^k\|_{2,D} = 0$$

and since $\|(T_{tt} - z)f_t^k\|_{W_2^{2s_t}} \to 0, k \to \infty$, then $\lim_{k\to\infty} \|(T_{tt} - z)S\overline{\partial}^i f_t^k\|_{2,D} = 0$. Now, from the fact that totally P-posinormal operators possess property (β)

(Lemma 4) follows that $S\overline{\partial}^i f_t^k \to \infty$, $i = 0, 1, ..., 2s_{t-1}$, uniformly on compact sets of D. Let K be a compact set such that $\sigma(T) \subset K \subset D$. Then

$$\begin{aligned}\|S\overline{\partial}^i f_t^k\|_{2,D}^2 &= \int_D \|S\overline{\partial}^i f_t^k(z)\|^2 dz \\ &= \int_K \|S\overline{\partial}^i f_t^k(z)\|^2 dz + \int_{D\setminus K} \|S\overline{\partial}^i f_t^k(z)\|^2 dz.\end{aligned}$$

The operator T is invertible on $D \setminus K$, therefore the second integral tends to 0. Then, from (8) we get $\|\overline{\partial}^i f_t^k\|_{2,D} \to 0$, $i = 0, 1, 2, ..., 2s_{t-1}$; hence $\|f_t^k\|_{2,D} \to 0$, as $k \to \infty$. Now, returning to the proof of (7) we consequently have

$$\begin{aligned}&\|(T_{jj}-z)f_j^k + T_{jj+1}f_{j+1}^k + \cdots + T_{jt-1}f_{t-1}^k + 1\otimes h_j^k\|_{W_2^{2s_{t-1}}} \\ = &\|(T_{jj}-z)f_j^k + T_{jj+1}f_{j+1}^k + \cdots + T_{jt-1}f_{t-1}^k + T_{jt}f_t^k - T_{jt}f_t^k + 1\otimes h_j^k\|_{W_2^{2s_{t-1}}} \\ \le &\|(T_{jj}-z)f_j^k + T_{jj+1}f_{j+1}^k + \cdots + T_{jt}f_t^k + 1\otimes h_t^k\|_{W_2^{2s_{t-1}}} + \|T_{jt}f_t^k\|_{W_2^{2s_{t-1}}} \\ \le &\|(T_{jj}-z)f_j^k + T_{jj+1}f_{j+1}^k + \cdots + T_{jt}f_t^k + 1\otimes h_t^k\|_{W_2^{2s_t}} + \|T_{jt}f_t^k\|_{W_2^{2s_{t-1}}}.\end{aligned}$$

Thus (7) is proved, which finishes the induction. Now, from Lemma 3 we get $h_i^k \to 0$, $i = 1, 2, ..., n$. The proof is completed. ■

Theorem 6. *Let*

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & T_{22} & \cdots & T_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & T_{nn} \end{pmatrix} \in \mathcal{L}(\oplus_1^n \mathcal{H}),$$

where T_{ii} are totally P-posinormal operators with corresponding polynomials $P_i(z)$ of degree $degP_i = m_i$, $i = 1, 2, ..., n$. Then the operator T is subscalar of order $2s$, where $s = \sum_{i=1}^n m_i$.

Proof. The statement of the theorem follows directly from Lemma 5 and equality (3). The proof is completed. ■

Since every hyponormal operator is totally P-posinormal (with polynomial $P(z) = z$ of degree 1 and $M(z) \equiv 1$) Theorem 6 generalizes the result of E. Ko ([1], Theorem 4.5).

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