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Bernstein-Stancu Operators on the Standard Simplex[1]

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In this paper we introduce a simplicial composition of two different positive approximation processes on a finite real interval in order to construct a positive approximation process on a d-dimensional simplex. We take into consideration the sequences of Bernstein and Stancu operators, but the same method can be applied to different sequences of positive operators. We can also obtain estimates of the rate of convergence and a Voronovskaja's formula for the compound operators. The application to the approximation of the solution of suitable parabolic problems constitutes one of the main motivation of this construction.

1. Introduction

Consider the standard simplex

$$K_d := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; x_1, \ldots, x_d \geq 0\,,\ \sum_{i=1}^{d} x_i \leq 1 \right\} \tag{1.1}$$

of $\mathbb{R}^d$.

Many examples of diffusion models are described in suitable function spaces on K_d. Particular interest has the study of the following evolution prob-

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$$(1.2)\quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \displaystyle\sum_{i,j=1}^{d} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) & \\ \qquad\qquad + \displaystyle\sum_{i=1}^{d} \beta_i(x)\frac{\partial u}{\partial x_i}(x,t), & x=(x_1,\dots,x_d)\in K_d\,, \\ & t>0\,, \\ u(x,0)=u_0(x)\,, & x\in K_d\,. \end{cases}$$

For example, the preceding problem is a gene frequency model in population genetics in the particular case where $\alpha_{ij}(x) = x_i(\delta_{ij} - x_j)/2$; in this case every vertex of K_d represents an allele. This diffusion process is known as Fleming-Viot model; there are different types of Fleming-Viot models and in order to approximate the solution of (1.2) we may need to combine different sequences of operators which take into account the different behavior of each allele.

More precisely, if we assign d sequences of approximation processes on $C[0,1]$, we define a corresponding sequence of linear operators on $C(K_d)$ and we study its approximation properties and the validity of a Voronovskaja's formula. Finally, connections with the evolution problem (1.2) are also considered.

For the sake of brevity, we limit ourselves to the leading case $d = 2$; the generalization to an arbitrary dimension is straightforward and for this reason we shall omit the details.

We shall deal with sequences $(L_n)_{n\geq 1}$ of linear operators on $C[0,1]$ having the form

$$L_n f(x) := \sum_{k=0}^{n} \alpha_{n,k}(x)\,\lambda_{n,k}(f)\,,$$

where, for every $n \geq 1$ and $k = 0,\dots,n$, the functions $\alpha_{n,k}$ are positive continuous functions on $[0,1]$ and $\lambda_{n,k} : C[0,1] \to \mathbb{R}$ are assigned Radon measures. Thus, for every $n \geq 1$, L_n is a positive linear operator.

The difficulty in extending the operators L_n to the simplex K_2 is mainly due to the fact that we have to point out separately the dependence of the functions $\alpha_{n,k}$ on the variables x and $1-x$ in order to extend these functions to K_2.

However, if we assume $\alpha_{n,k}(x) = a_{n,k}\varphi_k(x)\varphi_{n-k}(1-x)$, their extensions can be easily defined as

$$\alpha_{n,h,k}(x,y) := a_{n,h}a_{n-h,k}\varphi_h(x)\varphi_k(y)\varphi_{n-h-k}(1-x-y)$$

for every $h+k \leq n$.

As regards to the Radon measures $\lambda_{n,k}$ we need to extend them to Radon measures $\lambda_{n,h,k}$ on $C(K_2)$.

We observe that if λ and μ are Radon measures on $[0,1]$, we can consider the tensor product $\lambda \otimes \mu$ which is defined on $C([0,1]^2)$ as the unique Radon measure which takes the value

$$(1.3) \qquad (\lambda \otimes \mu)(f) := \lambda(\varphi) \cdot \mu(\psi)$$

for every $f(x,y) = \varphi(x)\,\psi(y)$, $(x, y \in [0,1])$, with $\varphi, \psi \in C[0,1]$. This follows from the density of these functions in $C([0,1]^2)$ with respect to the sup-norm (see, e.g., Choquet [4, Lemma 13.8]). Now, the same functions are dense in $C(K_2)$ too (indeed, every $f \in C(K_2)$ admits a continuous extension to $[0,1]^2$) and therefore (1.3) determines a unique Radon measure also on the simplex K_2.

However, for our purposes it will be more useful to define the *simplicial tensor product* $\lambda \otimes_s \mu$ of λ and μ as the unique extension obtained by setting

$$(\lambda \otimes_s \mu)(f) := \lambda(\varphi) \cdot \mu(\psi_a)\,, \qquad a = 1 - \max(\mathrm{supp}\,(\lambda))\,, \qquad \psi_a = \psi(a\cdot)$$

for every $f(x,y) = \varphi(x)\,\psi(y)$.

At this point, for every $h + k \le n$ we put $\lambda_{n,h,k} := \lambda_{n,h} \otimes_s \lambda_{n-h,k}$ and we can generalize the operators L_n to K_2 by setting, for every $f \in C(K_2)$ and $(x,y) \in K_2$,

$$L_n f(x,y) := \sum_{h=0}^{n} \sum_{k=0}^{n-h} \alpha_{n,h,k}(x,y) \lambda_{n,h,k}(f)\,.$$

If we apply the preceding definition to some classical approximation processes on $[0,1]$, we obtain their classical generalization to K_2. For example, consider the classical *Bernstein operators* $B_n : C[0,1] \to C[0,1]$ on the interval $[0,1]$ which are defined by setting, for every $f \in C[0,1]$ and $x \in [0,1]$,

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k\,(1-x)^{n-k} f\left(\frac{k}{n}\right)\,.$$

In this case, the extension to K_2 yields the operator $B_n : C(K_2) \to C(K_2)$ defined by

$$\begin{aligned} B_n f(x,y) \quad &:= \quad \sum_{h=0}^{n} \sum_{k=0}^{n-h} \binom{n}{h} \binom{n-h}{k} x^h\, y^k\, (1-x-y)^{n-h-k} \\ &\qquad \times f\left(\frac{h}{n}, \left(1 - \frac{h}{n}\right) \frac{k}{n-h}\right) \\ &= \quad \sum_{h+k \le n} \frac{n!}{h!\,k!\,(n-h-k)!} x^h\, y^k\, (1-x-y)^{n-h-k} f\left(\frac{h}{n}, \frac{k}{n}\right)\,, \end{aligned}$$

($f \in C(K_2)$, $(x,y) \in K_2$), which is the classical extension of B_n to K_2.

The advantage of the above procedure consists mainly in the fact that it can be applied even to different sequences of positive linear operators; so, if

$$M_n f(x) := \sum_{k=0}^{n} \beta_{n,k}(x)\, \mu_{n,k}(f)\,,$$

are positive linear operators with $\beta_{n,k}(x) = b_{n,k}\psi_k(x)\psi_{n-k}(1-x)$ we can define the *simplicial composition* $L_n \otimes_s M_n : C(K_2) \to C(K_2)$ of the sequences $(L_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ by setting, for every $f \in C(K_2)$ and $(x,y) \in K_2$,

$$(L_n \otimes_s M_n) f(x,y) := \sum_{h=0}^{n}\sum_{k=0}^{n-h} a_{n,h}\, b_{n-h,k}\, \varphi_h(x)\, \psi_k(y)\, \psi_{n-h-k}(1-x-y) \times (\lambda_{n,h} \otimes_s \mu_{n-h,k})(f)\,.$$

However, the above sequence has the disadvantage that even if $L_n(\mathbf{1}) = \mathbf{1}$ and $M_n(\mathbf{1}) = \mathbf{1}$ for every $n \geq 1$ we do not necessarily have $(L_n \otimes_s M_n)(\mathbf{1}) = \mathbf{1}$. For this reason, if we put

$$\chi_n(x,y) = \sum_{k=0}^{n} b_{n,k}\, \psi_k(x)\, \psi_{n-k}(y) \qquad (x,y) \in K_2\,,$$

it may be convenient to replace the preceding definition with the following

$$(L_n \otimes_s M_n) f(x,y) := \sum_{h=0}^{n}\sum_{k=0}^{n-h} a_{n,h}\, b_{n-h,k}\, \varphi_h(x)\, \frac{\varphi_{n-h}(1-x)}{\chi_{n-h}(y, 1-x-y)} \times \psi_k(y)\, \psi_{n-h-k}(1-x-y)\, (\lambda_{n,h} \otimes_s \mu_{n-h,k})(f)\,.$$

In this case, if $L_n\mathbf{1} = \mathbf{1}$ and $M_n\mathbf{1} = \mathbf{1}$ for every $n \geq 1$ and if the measures $\mu_{n,k}$ are probability Radon measures, then $(L_n \otimes_s M_n)\mathbf{1} = \mathbf{1}$ for every $n \geq 1$.

No matter of the generality of the preceding construction, in many cases the extensions of the functions $\alpha_{n,k}$ are quite natural and we can choose the factor $\chi_n(x,y)$ in a very suitable way.

In the sequel, we are interested to perform the above construction taking into consideration two classical sequences of positive operators, namely the sequences of Bernstein and of Stancu operators.

2. Bernstein-Stancu operators on K_2

The classical *Stancu operators* $S_{n,a_n} : C[0,1] \to C[0,1]$ on the interval $[0,1]$ are defined by setting, for every $f \in C[0,1]$ and $x \in [0,1]$,

$$S_{n,a_n} f(x) := \frac{1}{p_n(a_n)} \sum_{k=0}^{n} \binom{n}{k} \Phi_k(x; a_n)\, \Phi_{n-k}(1-x; a_n)\, f\left(\frac{k}{n}\right) ,$$

where $(a_n)_{n\in\mathbb{N}}$ is an assigned sequence of positive real numbers and

$$\Phi_k(y; a_n) := \prod_{j=0}^{k-1} (y + j a_n) , \qquad p_n(a_n) := \Phi_n(1; a_n)$$

with the convention, $\Phi_0(y; a_n) = 1$ (see also [12, 13]).

If no confusion arises, in the sequel we shall briefly write S_n in place of S_{n,a_n}.

A different expression of Stancu operators which is very useful in studying some of their approximation properties, has been obtained in [8, pp. 61–67] and [9] (see also [1, (6.3.21)]); indeed, for every $f \in C[0,1]$ and $x \in [0,1]$ we have

$$\begin{aligned} S_n f(x) &= \frac{1}{p_n(a_n)} \sum_{k=1}^{n} \frac{n!}{k!} a_n^{n-k} \sum_{|v|_k = n} \frac{1}{v_1 \cdots v_k} \\ &\quad \times \sum_{m=0}^{k} x^m (1-x)^{k-m} \sum_{\{i_1,\dots,i_m\} \in C(k,m)} f\left(\frac{v_{i_1} + \ldots + v_{i_m}}{n}\right), \end{aligned} \tag{2.1}$$

where the notation $|v|_k = n$ means that the sum is extended to all $(v_1, \ldots, v_k) \in \{1, \ldots, n\}^k$ such that $v_1 + \ldots + v_k = n$ and $C(k,m)$ denotes the set of all subsets of $\{1, \ldots, k\}$ having m different elements if $m \geq 1$, while if $m = 0$ we set $C(k,m) = \emptyset$ and $v_{i_1} + \ldots + v_{i_m} = 0$.

Inspired from the general procedure described in Section 1, we define the *Bernstein-Stancu operators* $B_n \otimes_s S_n : C(K_2) \to C(K_2)$ on the simplex K_2 by putting, for every $f \in C(K_2)$ and $(x,y) \in K_2$,

$$\begin{aligned} (B_n \otimes_s S_n) f(x,y) &:= \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x; a_{n-h})} \sum_{k=0}^{n-h} \binom{n-h}{k} \\ &\quad \times \Phi_k(y; a_{n-h})\, \Phi_{n-h-k}(1-x-y; a_{n-h})\, f\left(\tfrac{h}{n}, \tfrac{k}{n}\right) \\ &= \sum_{h+k \leq n} \frac{n!}{h!\, k!\, (n-h-k)!} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x; a_{n-h})} \\ &\quad \times \Phi_k(y; a_{n-h})\, \Phi_{n-h-k}(1-x-y; a_{n-h})\, f\left(\tfrac{h}{n}, \tfrac{k}{n}\right) \end{aligned} \tag{2.2}$$

The factor $(1-x)^{n-h}/\Phi_{n-h}(1-x;a_{n-h})$ is needed in order to simplify the following formulas.

Bernstein-Stancu operators have also an interesting probabilistic interpretation. Indeed, according to the fact that Bernstein operators are associated with a Polya urn scheme with two different objects and without replacement while Stancu operators are associated with a similar scheme where at the n-th step the extracted object is replaced by a_n objects of the same type, these operators are related to an urn scheme with three objects and where only two are replaced with a_n objects of the same type.

Of course, if $a_j = 0$, $j = 1, \dots, n$, we have $S_j = B_j$ for every $j = 1, \dots, n$ and consequently $B_n \otimes_s S_n$ coincides with the classical n-th Bernstein operator on the two dimensional simplex; referring to the above probabilistic interpretation, the replacement with $a_j = 0$ objects means that there is no replacement.

Using (2.1), we obtain, for every $f \in C(K_2)$ and $(x,y) \in K_2$,

$$\begin{aligned}(B_n \otimes_s S_n) f(x,y) &= \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!} \\ &\times a_{n-h}^{n-h-k} \sum_{|v|_k = n-h} \frac{1}{v_1 \cdots v_k} \sum_{m=0}^{k} y^m (1-x-y)^{k-m} \\ &\times \sum_{\{i_1,\dots,i_m\} \in C(k,m)} f\left(\frac{h}{n}, \frac{v_{i_1} + \dots + v_{i_m}}{n}\right)\end{aligned}$$

where, by convention, empty sums are equal to 0 (these occur in the variable k if $h = n$).

We recall that (see e.g. [8, (1.1.5), p. 13 or (1.1.9), p. 14] or [1, (6.1.25) and (6.1.26)])

$$p_n(a_n) = \sum_{k=1}^{n} \frac{n!}{k!} a_n^{n-k} \sum_{|v|_k = n} \frac{1}{v_1 \cdots v_k}$$

and hence (see also [8, (3.2.26), p. 66])

$$\Phi_n(x;a_n) = \sum_{k=1}^{n} \frac{n!}{k!} a_n^{n-k} x^k \sum_{|v|_k = n} \frac{1}{v_1 \cdots v_k}. \tag{2.3}$$

Taking this into account, for every $(x,y) \in K_2$ we obtain

$$(B_n \otimes_s S_n) \mathbf{1}(x,y) = \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!}$$

$$\times a_{n-h}^{n-h-k} \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k} \sum_{m=0}^{k} \binom{k}{m} y^m (1-x-y)^{k-m}$$

$$= \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!}$$

$$\times a_{n-h}^{n-h-k}(1-x)^k \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k}$$

$$= \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} = 1\,.$$

Analogously, it follows

$$\begin{aligned}(B_n \otimes_s S_n)\mathrm{pr}_1(x,y) &= \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!} \\ &\quad \times a_{n-h}^{n-h-k} \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k} \\ &\quad \times \sum_{m=0}^{k} \binom{k}{m} y^m (1-x-y)^{k-m} \frac{h}{n} \\ &= \sum_{h=0}^{n} \binom{n}{h} \frac{h}{n} x^h (1-x)^{n-h} \\ &= x \sum_{h=1}^{n} \binom{n-1}{h-1} x^{h-1} (1-x)^{(n-1)-(h-1)} = x\end{aligned}$$

and, taking into account the formula

$$\sum_{\{i_1,\ldots,i_m\}\in C(k,m)} (v_{i_1} + \ldots + v_{i_m}) = (n-h) \binom{k-1}{m-1}$$

which can be easily derived by finite induction on $m = 1, \ldots, k$ (with the convention that the sum is equal to 0 if $m = 0$), we also have

$$\begin{aligned}&(B_n \otimes_s S_n)\mathrm{pr}_2(x,y) \\ &= \sum_{h=0}^{n} \binom{n}{h} \frac{n-h}{n} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!} \\ &\quad \times a_{n-h}^{n-h-k} \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k} \sum_{m=1}^{k} \binom{k-1}{m-1} y^m (1-x-y)^{k-m}\end{aligned}$$

$$
\begin{aligned}
&= y \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \frac{n-h}{n} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!} a_{n-h}^{n-h-k} \\
&\quad \times (1-x)^{k-1} \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k} \\
&= y \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \frac{n-h}{n} \frac{1}{1-x} = y
\end{aligned}
$$

(if $x = 1$ the preceding formula can be verified directly).

At this point, we evaluate the operators $B_n \otimes_s S_n$ at the functions $\mathrm{pr}_i \mathrm{pr}_j$, $i, j = 1, 2$. For the sake of brevity, we omit some intermediate equalities which are similar to the preceding ones.

We have

$$
\begin{aligned}
&(B_n \otimes_s S_n)(\mathrm{pr}_1^2)(x,y) \\
&\quad = \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!} \\
&\quad\quad \times a_{n-h}^{n-h-k} \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k} \sum_{m=0}^{k} \binom{k}{m} y^m (1-x-y)^{k-m} \frac{h^2}{n^2} \\
&\quad = \sum_{h=0}^{n} \binom{n}{h} \frac{h^2}{n^2} x^h (1-x)^{n-h} = x^2 + \frac{x(1-x)}{n}
\end{aligned}
$$

and further

$$
\begin{aligned}
&(B_n \otimes_s S_n)(\mathrm{pr}_1 \mathrm{pr}_2)(x,y) \\
&\quad = \sum_{h=0}^{n} \binom{n}{h} \frac{n-h}{n} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \\
&\quad\quad \times \sum_{k=1}^{n-h} \frac{(n-h)!}{k!} a_{n-h}^{n-h-k} \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k} \\
&\quad\quad \times \sum_{m=1}^{k} \binom{k-1}{m-1} \frac{h}{n} y^m (1-x-y)^{k-m} \\
&\quad = y \sum_{h=0}^{n} \binom{n}{h} x^h \frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})} \frac{h}{n} \frac{n-h}{n} \\
&\quad\quad \times \sum_{k=1}^{n-h} \frac{(n-h)!}{k!} a_{n-h}^{n-h-k} \times (1-x)^{k-1} \sum_{|v|_k=n-h} \frac{1}{v_1 \cdots v_k}
\end{aligned}
$$

$$= y\sum_{h=0}^{n}\binom{n}{h} x^h(1-x)^{n-h}\frac{h}{n}\left(1-\frac{h}{n}\right)\frac{1}{1-x}$$
$$= \frac{y}{1-x}\left(x-x^2-\frac{x(1-x)}{n}\right)=\frac{n-1}{n}xy$$

(again, the equality can be proved directly at $x=1$).

Finally, using the formula (see, e.g., [8, (1.1.10)] or also [2, (2.9)] and [1, Proposition 6.1.5, (6.1.33), pp. 386–7])

$$p_{n+1}(a)=p_2(a)\sum_{k=1}^{n}\frac{(n-1)!}{k!}a^{n-k}\sum_{|v|_k=n}\frac{v_1^2+\ldots+v_k^2}{v_1\cdots v_k},$$

we can evaluate $(B_n\otimes_s S_n)(\mathrm{pr}_2^2)$ as in [2, (2.17)] and obtain

$$\begin{aligned}(B_n\otimes_s S_n)(\mathrm{pr}_2^2)(x,y)&\\ &= \sum_{h=0}^{n}\binom{n}{h}x^h\frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x;a_{n-h})}\\ &\quad\times\left(y(1-y)\frac{1+(n-h)\,a_{n-h}}{n\,(1+a_{n-h})}+y^2\,\Phi_{n-h}(1-x;a_{n-h})\right)\\ &= y^2+y(1-y)\sum_{h=0}^{n}\binom{n}{h}x^h(1-x)^{n-h}\frac{1+(n-h)\,a_{n-h}}{n\,(1+a_{n-h})}\\ &= y^2+\frac{y(1-y)}{n}\sigma_n(x),\end{aligned}$$

where we have put, for simplicity,

$$\sigma_n(x):=\sum_{h=0}^{n}\binom{n}{h}x^h(1-x)^{n-h}\frac{1+(n-h)\,a_{n-h}}{1+a_{n-h}}.$$

It is well-known that Stancu operators converge strongly to the identity operator if the sequence $(a_n)_{n\geq 1}$ tends to 0. Now, we prove the analogous result for the Bernstein-Stancu operators. First of all, we establish the following lemma.

Lemma 2.1. *Assume that the sequence $(a_n)_{n\geq 1}$ tends to 0. Then, the sequence of functions*

$$g_n(x):=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\frac{k}{n}a_k$$

converges uniformly to 0 on the interval $[0,1]$.

Proof. Let $\varepsilon > 0$ and let $p \geq 1$ such that $a_n \leq \varepsilon$ for every $n \geq p$. Denote by M the maximum of the sequence $(a_n)_{n\geq 1}$ and put

$$\delta := \frac{\varepsilon}{M}\,, \qquad \nu := \min\left\{n \geq 1 \mid n \geq \frac{p}{\delta}\right\} .$$

For every $n \geq \nu$ and $x \in [0,1]$, we have

$$\begin{aligned} g_n(x) &= \sum_{k=0}^{p} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} a_k + \sum_{k=p+1}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} a_k \\ &\leq M \sum_{k=0}^{p} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} + \varepsilon . \end{aligned}$$

Now, we observe that for every $k = 0, \dots, p$ the function $x^k(1-x)^{n-k}$ attains its maximum at the point $k/n \leq \delta$ and is decreasing on the interval $[\delta, 1]$. It follows that also the maximum of $\sum_{k=0}^{p} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n}$ is achieved in $[0,\delta]$. Since for every $x \in [0,\delta]$

$$g_n(x) \leq M \sum_{k=0}^{p} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} + \varepsilon \leq Mx + \varepsilon \leq M\delta + \varepsilon = 2\varepsilon\,,$$

the proof is complete. ∎

We observe that $1/(1+a_{n-h}) \leq 1$ and therefore, for every $(x,y) \in K_2$,

$$\begin{aligned} |(B_n \otimes_s S_n)(\mathrm{pr}_2^2)(x,y) - y^2| &= y(1-y)\Bigg(\frac{1}{n} \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \frac{1}{1+a_{n-h}} \\ &\quad + \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \frac{(n-h)\, a_{n-h}}{n\,(1+a_{n-h})}\Bigg) \\ &\leq y(1-y)\left(\frac{1}{n} + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (1-x)^k \frac{k}{n} a_k\right) \\ &= y(1-y)\left(\frac{1}{n} + g_n(1-x)\right) . \end{aligned}$$

Hence, under the assumption that $(a_n)_{n\geq 1}$ tends to 0, from the prececeding lemma we deduce that $((B_n \otimes_s S_n)(\mathrm{pr}_2^2))_{n\geq 1}$ converges uniformly to the function pr_2^2.

At this point, we can state the following approximation properties of the sequence $(B_n \otimes_s S_n)_{n\geq 1}$.

In the sequel, we denote by $T : C(K_2) \to C(K_2)$ the standard projection which maps every $f \in C(K_2)$ onto the unique affine function which interpolates f at the vertices of K_2. Thus, for every $f \in C(K_2)$ and $(x, y) \in K_2$,

$$Tf(x,y) = xf(1,0) + yf(0,1) + (1 - x - y)f(0,0) .$$

Moreover, we define the iterates of $B_n \otimes_s S_n$ by setting, as usual

$$(B_n \otimes_s S_n)^1 = B_n \otimes_s S_n , \quad (B_n \otimes_s S_n)^{m+1} = (B_n \otimes_s S_n)^m \circ (B_n \otimes_s S_n) , \quad m \geq 1 .$$

Theorem 2.2. *Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers. We have the following properties:*

1. *For every $n \geq 1$ and $f \in C(K_2)$,*

$$\lim_{m\to+\infty} (B_n \otimes_s S_n)^m f = T(f) \qquad \text{uniformly on } K_2 .$$

2. *If the sequence $(a_n)_{n\geq 1}$ tends to 0, then for every $m \geq 1$ and $f \in C(K_2)$,*

$$\lim_{n\to+\infty} (B_n \otimes_s S_n)^m f = f \qquad \text{uniformly on } K_2 .$$

P r o o f. From the preceding formulas, we obtain in a straightforward way, for every $n, m \geq 1$, $f \in C(K_2)$ and $(x, y) \in K_2$,

$$\begin{aligned}
(B_n \otimes_s S_n)^m \mathbf{1}(x,y) &= 1 , \\
(B_n \otimes_s S_n)^m \mathrm{pr}_1(x,y) &= x , \\
(B_n \otimes_s S_n)^m \mathrm{pr}_2(x,y) &= y , \\
(B_n \otimes_s S_n)^m (\mathrm{pr}_1^2)(x,y) &= \left(1 - \frac{1}{n}\right)^m x^2 + \left(1 - \left(1 - \frac{1}{n}\right)^m\right) x ; \\
(B_n \otimes_s S_n)^m (\mathrm{pr}_2^2)(x,y) &= \left(1 - \frac{\sigma_n(x)}{n}\right)^m y^2 + \left(1 - \left(1 - \frac{\sigma_n(x)}{n}\right)^m\right) y
\end{aligned}$$

and therefore the first part of the theorem follows from [1, Example 3.3.5, p. 173] and the second part from Volkov's theorem (see, e.g., [1, Example 2 to Theorem 4.4.6, p. 245]). ∎

Using the same methods of [2, Section 3, Theorem 3.4], we can also obtain a quantitative estimate of the convergence. In our case, for every $f \in C(K_2)$, we have (see [2, Definition 3.1])

$$\|(B_n \otimes_s S_n)f - f\| \leq \left(1 + \frac{\|\sigma_n\|}{n}\right) \Omega\left(f, \frac{1}{\sqrt{n}}\right) .$$

Remark 2.3. Among the general properties of the Bernstein-Stancu operators, we can also easily obtain the preservation of the k-convexity and the Lipschitz class $\mathrm{Lip}_M 1$, $M > 0$.

Indeed, it is well-known that the Stancu operator S_{n,a_n} preserves the k-convexity (see [10, 11]) and the Lipschitz class $\mathrm{Lip}_M 1$ (see [5]).

Now, for every $f \in C(K_2)$ and $h = 0, \ldots, n$, let

$$\varphi_{n,h}(t) := f\left(\frac{h}{n}, \frac{n-h}{n}t\right), \qquad 0 \le t \le 1 .$$

Then, according to (2.2),

$$(B_n \otimes_s S_{n,a_n})f(x,y) = \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \left(S_{n-h,a_{n-h}/(1-x)}\varphi_{n,h}\right)\left(\frac{y}{1-x}\right)$$

for every $(x,y) \in K_2 \setminus \{(1,0)\}$.

Hence, it easily follows that if $f(a,\cdot)$ is k-convex (respectively, in $\mathrm{Lip}_M 1$, $M > 0$) on $[0, 1-a]$ for all $a \in [0,1[$, then $(B_n \otimes_s S_{n,a_n})f(x,\cdot)$ is k-convex (respectively, in $\mathrm{Lip}_M 1$) on $[0, 1-x]$ for all $x \in [0,1[$.

At this point, we conclude our investigation of Bernstein-Stancu operators stating a Voronovskaja's type formula for these operators. The guidelines are the methods used in [3] and [1, Theorem 6.2.5, p. 433], but we need some supplementary tools.

Lemma 2.4. *If $(a_n)_{n\ge1}$ is a sequence of positive real numbers converging to 0, we have*

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} a_k = 0$$

uniformly on every interval $[\delta, 1]$ with $0 < \delta < 1$ (we use as usual the convention $a_0 = 0$).

Proof. Fix $\delta > 0$ and let $\varepsilon > 0$. Then, there exists $p \ge 1$ such that $a_n \le \varepsilon$ for every $n \ge p$. Now, let $n \ge p/\delta$; we have

$$\frac{k}{n} \le \frac{p}{n} \le \delta \qquad \text{for every } k = 0, \ldots, p$$

and therefore

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} a_k = \sum_{k=0}^{p} \binom{n}{k} x^k (1-x)^{n-k} a_k$$

$$+\sum_{k=p+1}^{n} \binom{n}{k} x^k (1-x)^{n-k} a_k$$
$$\leq M \sum_{k=0}^{p} \binom{n}{k} x^k (1-x)^{n-k} + \varepsilon ,$$

where $M := \sup_{k\geq 1} a_k$. As regards to the sum $\sum_{k=0}^{p} \binom{n}{k} x^k (1-x)^{n-k}$, we observe that the maximum of each addend $x^k (1-x)^{n-k}$ is attained in $k/n \in [0, \delta]$ and therefore each addend is decreasing in $[\delta, 1]$.

Thus, the preceding inequalities yield

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} a_k \leq M \sum_{k=0}^{p} \binom{n}{k} \delta^k (1-\delta)^{n-k} + \varepsilon ,$$

and since $\lim_{n\to+\infty} \sum_{k=0}^{p} \binom{n}{k} \delta^k (1-\delta)^{n-k} = 0$, the proof is complete. ∎

Theorem 2.5. *Assume that the sequence $(n \cdot a_n)_{n\geq 1}$ converges to a positive real number b.*

Then, for every $u \in C^2(K_2)$, we have

$$\lim_{n\to+\infty} n \left((B_n \otimes_s S_n)u(x,y) - u(x,y)\right)$$
$$= \frac{x(1-x)}{2} \frac{\partial^2 u}{\partial x^2}(x,y) + (1+b) \frac{y(1-y)}{2} \frac{\partial^2 u}{\partial y^2}(x,y) - xy \frac{\partial^2 u}{\partial x \, \partial y}(x,y)$$

uniformly on K_2.

Proof. First, we show that the sequence $(\sigma_n(x))_{n\geq 1}$ converges to $1+b$ uniformly on every interval $[0,\delta]$ with $0 < \delta < 1$. Indeed, for every $x \in [0,\delta]$, we have

$$\begin{aligned}
|\sigma_n(x) - (1+b)| &= \left|\sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \left(\frac{1+(n-h)a_{n-h}}{1+a_{n-h}} - (1+b)\right)\right| \\
&= \left|\sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \frac{(n-h)a_{n-h} - b - (1+b)\, a_{n-h}}{1+a_{n-h}}\right| \\
&\leq \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \left|(n-h)a_{n-h} - b\right| \\
&\quad +(1+b) \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} a_{n-h}
\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (1-x)^k x^{n-k} |ka_k - b| + (1+b) \sum_{k=0}^{n} \binom{n}{k} (1-x)^k x^{n-k} a_k \, .$$

Since the sequences $(|na_n - b|)_{n\geq 1}$ and $(a_n)_{n\geq 1}$ converge to 0, from the preceding lemma we have the convergence of the sequence $(\sigma_n(x))_{n\geq 1}$.

As a consequence of the above property, we have that the sequence $(y(1-y)\,\sigma_n(x))_{n\geq 1}$ converges uniformly to $(1+b)\,y(1-y)$ on K_2. To show this, fix $\varepsilon > 0$ and denote by c the supremum of the sequence $(|\sigma_n(x) - (1+b)|)_{n\geq 1}$ with respect to $n \geq 1$ and $x \in [0,1]$ (observe that the sequence is bounded at the point 1). Then, we can find $\nu \geq 1$ such that $|\sigma_n(x) - (1+b)| < \varepsilon$ for every $n \geq \nu$ and $x \in [0, 1-\varepsilon/c]$. Hence, for every $n \geq \nu$ and $(x,y) \in K_2$ we have $(y(1-y)\,|\sigma_n(x) - (1+b)| < \varepsilon$ if $x \leq 1 - \varepsilon/c$, while if $x > 1 - \varepsilon/c$ we have $y \leq 1 - x < \varepsilon/c$ and consequently $y(1-y)\,|\sigma_n(x) - (1+b)| < (\varepsilon/c)\,c = \varepsilon$. Since $\varepsilon > 0$ is arbitrarily chosen, the uniform convergence on K_2 has been proved.

Now, let $u \in C^2(K_2)$; for every (s,t), $(x,y) \in K_2$, we can write

$$\begin{aligned} u(s,t) &= u(x,y) + \frac{\partial u}{\partial x}(x,y)\,(s-x) + \frac{\partial u}{\partial y}(x,y)\,(t-y) \\ &+ \frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}(x,y)\,(s-x)^2 + \frac{1}{2}\,\frac{\partial^2 u}{\partial y^2}(x,y)\,(t-y)^2 \\ &+ \frac{\partial^2 u}{\partial x\,\partial y}(x,y)\,(s-x)\,(t-y) \\ &+ \omega((s,t),(x,y))\,\left((s-x)^2 + (t-y)^2\right) , \end{aligned}$$

where $\omega : K_2 \times K_2 \to \mathbb{R}$ satisfies

$$(2.4) \qquad |\omega| \leq M \,, \qquad \lim_{(s,t)\to(x,y)} \omega((s,t),(x,y)) = 0 \quad \text{uniformly on } K_2$$

(see also [1, (1), p. 433 and (2)–(3), p. 434]).

Since

$$\begin{aligned} n\,((B_n \otimes_s S_n)u(x,y) - u(x,y)) &= n\,\frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}(x,y)\,(B_n \otimes_s S_n)(\mathrm{pr}_1 - x)^2(x,y) \\ &+ n\,\frac{1}{2}\,\frac{\partial^2 u}{\partial y^2}(x,y)\,(B_n \otimes_s S_n)(\mathrm{pr}_2 - y)^2(x,y) \\ &+ \frac{\partial^2 u}{\partial x\,\partial y}(x,y)\,(B_n \otimes_s S_n)((\mathrm{pr}_1 - x)\,(\mathrm{pr}_2 - y))(x,y) \end{aligned}$$

$$
\begin{aligned}
&+n\,(B_n \otimes_s S_n)\left(\omega(\cdot,(x,y))\ \left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)\right)(x,y)\\
&= n\,\frac{1}{2}\,\frac{x(1-x)}{n}\,\frac{\partial^2 u}{\partial x^2}(x,y) + n\,\frac{1}{2}\,\frac{y(1-y)}{n}\,\sigma_n(x)\,\frac{\partial^2 u}{\partial y^2}(x,y)\\
&-n\,\frac{xy}{n}\,\frac{\partial^2 u}{\partial x\,\partial y}(x,y)\\
&+(B_n \otimes_s S_n)\left(\omega(\cdot,(x,y))\ \left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)\right)(x,y)\\
&= \frac{x(1-x)}{2}\,\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{y(1-y)}{2}\,\sigma_n(x)\,\frac{\partial^2 u}{\partial y^2}(x,y) - xy\,\frac{\partial^2 u}{\partial x\,\partial y}(x,y)\\
&+(B_n \otimes_s S_n)\left(\omega(\cdot,(x,y))\ \left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)\right)(x,y)\,,
\end{aligned}
$$

the proof is complete if we show that

$$
\lim_{n\to+\infty} n\,(B_n \otimes_s S_n)\left(\omega(\cdot,(x,y))\ \left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)\right)(x,y) = 0
$$

uniformly on K_2.

Let $\varepsilon > 0$; from (2.4) we can find $\delta > 0$ such that $|\omega((s,t),(x,y))| \le \varepsilon$ for every $(s,t),(x,y) \in K_2$ such that $(x-s)^2 + (y-t)^2 \le \delta^2$.

As a consequence, we show that for every $(s,t) \in K_2$

$$
\begin{aligned}
&|\omega((s,t),(x,y))\ ((\mathrm{pr}_1(s,t) - x)^2 + (\mathrm{pr}_2(s,t) - y)^2)|\\
&\qquad \le \varepsilon\left((s-x)^2 + (t-y)^2\right) + \frac{2M}{\delta^4}\left((s-x)^2 + (t-y)^2\right)^2 .
\end{aligned}
\tag{2.5}
$$

Indeed, if $(s,t) \in K_2$ satisfies $(x-s)^2 + (y-t)^2 \le \delta^2$, we have

$$
\left|\omega((s,t),(x,y))\ \left((\mathrm{pr}_1(s,t) - x)^2 + (\mathrm{pr}_2(s,t) - y)^2\right)\right| \le \varepsilon\left((s-x)^2 + (t-y)^2\right)\,,
$$

while, if $(x-s)^2 + (y-t)^2 > \delta^2$, we have (see (2.4))

$$
\begin{aligned}
\left|\omega((s,t),(x,y))\ \left((\mathrm{pr}_1(s,t) - x)^2 + (\mathrm{pr}_2(s,t) - y)^2\right)\right| &\le 2M\\
&\le \frac{2M}{\delta^4}\left((s-x)^2 + (t-y)^2\right)^2 .
\end{aligned}
$$

At this point, we denote by a the supremum of $(n \cdot a_n)_{n\geq 1}$ and we have

$$
\begin{aligned}
&n\,(B_n \otimes_s S_n)\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)(x,y) \\
&\quad = x(1-x) + y(1-y)\,\sigma_n(x) \\
&\quad \leq x(1-x) + y(1-y) \sum_{h=0}^{n} \binom{n}{h} x^h\,(1-x)^{n-h}\,\frac{1+a}{1+a_{n-h}} \\
&\quad \leq x(1-x) + y(1-y)\,(1+a) \\
&\quad \leq \frac{1}{4} + \frac{1}{4}\,(1+a) \\
&\quad = \frac{1}{4}\,(2+a) =: c\,.
\end{aligned}
\tag{2.6}
$$

In order to estimate the last term in (2.5), we consider $i, j \in \{1,2\}$ and put $\tilde{h}_1 := \mathrm{pr}_i - \mathrm{pr}_i(x,y)$ and $\tilde{h}_2 := \mathrm{pr}_j - \mathrm{pr}_j(x,y)$. Following the proof of [1, Theorem 6.2.1, p. 424 and Lemma 6.2.2, p. 429], we can write

$$
\begin{aligned}
&n\,(B_n \otimes_s S_n)\left(\tilde{h}_1^2 \cdot \tilde{h}_2^2\right)(x,y) \\
&= \sum_{h=0}^{n} \binom{n}{h} x^h \,\frac{(1-x)^{n-h}}{\Phi_{n-h}(1-x; a_{n-h})} \sum_{k=1}^{n-h} \frac{(n-h)!}{k!}\, a_{n-h}^{n-h-k} \\
&\quad \times \sum_{|v|_k = n-h} \frac{(n-h)(1-x)^k}{v_1 \cdots v_k} \sum_{m=1}^{s_h} R_m(v_1, \ldots, v_k, v_{k+1})\, T\left(\tilde{h}_1^2 \cdot \tilde{h}_2^2\right)(x,y)
\end{aligned}
$$

where, by convention, $v_{k+1} := h$, s_h are suitable natural numbers and, for every $m = 1, \ldots, s_h$, there exist constants $c_m, d_m \in \mathbb{R}$ such that

$$
|R_m(v_1, \ldots, v_k, v_{k+1})| \leq \frac{1}{n^3}\, c_m \sum_{p=1}^{k} v_p^3 + \frac{1}{n^4}\, d_m \sum_{\{p_1,p_2\} \in C(k,2)} v_{p_1}^2 v_{p_2}^2\,.
$$

Finally, with the same arguments used in the proof of [1, Lemma 6.2.2, p. 429] and taking into account formula (2.3), we obtain

$$
\begin{aligned}
n\,(B_n \otimes_s S_n)\left(\tilde{h}_1^2 \cdot \tilde{h}_2^2\right)(x,y) &\leq \sum_{h=0}^{n} \binom{n}{h} x^h\,(1-x)^{n-h} \left(\frac{Cn+D}{n^2} \|\tilde{h}_1^2\|\,\|\tilde{h}_2^2\|\right) \\
&= \frac{Cn+D}{n^2} \|\tilde{h}_1^2\|\,\|\tilde{h}_2^2\|
\end{aligned}
$$

and hence

$$
n\,(B_n \otimes_s S_n)\left(\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)^2\right)(x,y)
$$

$$= \sum_{i,j=1}^{2} n\,(B_n \otimes_s S_n)\left(\left((\mathrm{pr}_i - \mathrm{pr}_i(x,y))^2 \cdot (\mathrm{pr}_j - \mathrm{pr}_j(x,y))^2\right)^2\right)(x,y)$$

$$\leq 4\,\frac{Cn + D}{n^2} \tag{2.7}$$

At this point, comparing (2.5) with (2.6) and (2.7), we obtain

$$n\left|(B_n \otimes_s S_n)\left(\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)^2\right)(x,y)\right| \leq c\varepsilon + 8\,\frac{M\,(Cn + D)}{\delta^4\, n^2}$$

and this completes the proof. ∎

The preceding Voronovskaja's type formula allows us to approximate the solution of the following evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,y,t) = \dfrac{x(1-x)}{2}\,\dfrac{\partial^2 u}{\partial x^2}(x,y,t) \\ \qquad\qquad +(1+b)\,\dfrac{y(1-y)}{2}\,\dfrac{\partial^2 u}{\partial y^2}(x,y,t) \\ \qquad\qquad -xy\,\dfrac{\partial^2 u}{\partial x\,\partial y}(x,y,t)\,, & (x,y) \in K_2\,,\ t > 0\,, \\ u(x,y,0) = u_0(x,y)\,, & (x,y) \in K_2\,, \end{cases} \tag{2.8}$$

where $u_0 \in C(K_2)$ and $b \in [0, +\infty[$ are assigned.

Indeed, consider the second-order differential operator

(2.9)
$$Au(x,y) = \frac{x(1-x)}{2}\,\frac{\partial^2 u}{\partial x^2}(x,y) + (1+b)\,\frac{y(1-y)}{2}\,\frac{\partial^2 u}{\partial y^2}(x,y) - xy\,\frac{\partial^2 u}{\partial x\,\partial y}(x,y),$$

defined for $u \in C^2(K_2)$.

A result by Ethier [6] establishes that the closure of A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $C(K_2)$ (see also [7]). Indeed, in this case the coefficients of the first derivatives terms are equal to 0 and therefore satisfy Ethier's condition; moreover the coefficient $1 + b$ does not influence the generation of a C_0-semigroup since it is a constant factor. (We point out that the generation of a C_0-semigroup is established with different methods also in [1, Theorem 6.2.6, p. 436].)

Hence, we can apply a theorem of Trotter [14] and represent the semigroup in terms of iterates of the operators $B_n \otimes_s S_n$ (see also [1, Chapter 6] for more details on this procedure).

Thus, for every $t \geq 0$ and every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have

$$\lim_{n\to+\infty} (B_n \otimes_s S_n)^{k(n)} = T(t) \qquad \text{strongly on } C(K_2)$$

and hence the solution of problem (2.8) is given by

$$u(x,y,t) = T(t)(u_0)(x,y) = \lim_{n\to+\infty} \left((B_n \otimes_s S_n)^{k(n)}(u_0)\right)(x,y) .$$

Finally, we point out that our construction can be applied even to sequences of Stancu operators associated with different sequences of positive real numbers, yielding in this case a new generalization of Stancu operators on K_2. Namely, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are assigned sequences of positive real numbers, we may define

$$\begin{aligned}(S_{n,a_n} \otimes_s S_{n,b_n})f(x,y) \quad := \quad & \sum_{h+k\leq n} \frac{n!}{h!k!(n-h-k)!}\Phi_h(x;a_n)\Phi_{n-h}(1-x;a_n) \\ & \times \frac{\Phi_k(y;b_n)\Phi_{n-h-k}(1-x-y;b_n)}{\Phi_{n-h}(1-x;b_{n-h})} f\left(\frac{h}{n},\frac{k}{n}\right) .\end{aligned}$$

The convergence properties of the sequence $(S_{n,a_n} \otimes_s S_{n,b_n})_{n\geq 1}$ can be investigated with the same techniques used above and for this reason we omit the details.

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