Mathematatica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 17, 2003, Fasc. 3-4

On Approximation by Bernstein-Chlodowsky Polynomials

E. İbikli

Presented by V. Kiryakova

The weighted approximation of continuous functions by Bernstein-Chlodowsky polynomials and their generalizations are studied.

AMS Subj. Classification: 41A36 , 41A35, 41A25
Key Words: approximation, Bernsteine-type polynomials

1. Introduction

The classical Bernstein-Chlodowsky polynomials have the following form

$$B_n(f,x)=\sum_{k=0}^{n} f(\frac{k}{n}b_n)C_n^k(\frac{x}{b_n})^k(1-\frac{x}{b_n})^{n-k}, \tag{1.1}$$

where $0 \le x \le b_n$ and b_n is the sequence of positive numbers such that $\lim\limits_{n\to\infty} b_n=\infty$, $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$. These polynomials were introduced by Chlodowsky in 1932 as a generalization of Bernstein polynomials (1912) on an unbounded set. Although there have been many studies of Bernstein polynomials to the present date (see [1], [2], [6] and [7]), the Bernstein-Chlodowsky polynomials (1.1) have not been investigated well enough. The aim of this article is to investigate the problem of weighted approximations of continuous functions by Bernstein-Chlodowsky polynomials (1.1) (for a generalization of these polynomials see [4]).

2. Main results

Let $\phi(x)$ be a continuous and increasing function in $(-\infty,\infty)$ such that

$$\lim_{x\to\pm\infty}\phi(x)=\pm\infty$$

and

$$\rho(x) = 1 + \phi^2(x)\,.$$

Denote by C_ρ the space of all continuous functions f, satisfying the condition

$$|f(x)| \le M_f \rho(x)\,, \qquad -\infty < x < \infty\,.$$

Obviously C_ρ is a linear normed space with the norm

$$\|f\|_\rho = \sup_{-\infty<x<\infty} \frac{|f(x)|}{\rho(x)}\,.$$

A Korovkin type theorem for linear positive operators L_n, acting from C_ρ to C_ρ, has been proved in [3], where the following results have been established.

Theorem 1. (See [3]) *There exists a sequence of positive linear operators L_n, acting from C_ρ to C_ρ, satisfying the conditions*

$$\lim_{n\to\infty} \|L_n(1,x) - 1\|_\rho = 0 \tag{2.1}$$

$$\lim_{n\to\infty} \|L_n(\phi,x) - \phi\|_\rho = 0 \tag{2.2}$$

$$\lim_{n\to\infty} \|L_n(\phi^2,x) - \phi^2\|_\rho = 0 \tag{2.3}$$

and there exists a function $f^* \in C_\rho$ for which

$$\overline{\lim_{n\to\infty}} \|L_n f^* - f^*\|_\rho > 0.$$

Theorem 2. (See [3]) *The conditions (2.1), (2.2), (2.3) imply $\lim_{n\to\infty} \|L_n f - f\|_\rho = 0$ for any function f belonging to the subset of C_ρ^0 of all functions $f \in C_\rho$ for which*

$$\lim_{|x|\to\infty} \frac{f(x)}{\rho(x)}$$

exists finitely.

Setting $\rho(x) = 1 + x^2$ and applying Theorem 2 to the operators

$$L_n(f,x) = \begin{cases} B_n(f,x) & \text{if } \ 0 \le x \le b_n \\ f(x) & \text{if } \ x \notin [0,b_n] \end{cases}$$

we obtain,

Proposition 3. *The assertion*

$$\lim_{n\to\infty} \sup_{0\le x\le b_n} \frac{|B_n(f,x)-f(x)|}{1+x^2}=0 \tag{2.4}$$

holds for any function $f\in C^0_\rho$, *with* $\rho(x)=1+x^2$, $x\ge 0$.

Note that conditions (2.1),(2.2) and (2.3) are fulfilled since

$$B_n(1,x)=1 \tag{2.5}$$

$$B_n(t,x)=x \tag{2.6}$$

$$B_n(t^2,x)=x^2+\frac{x(b_n-x)}{n} \tag{2.7}$$

and therefore

$$\sup_{0\le x\le b_n}\frac{|B_n(t^2,x)-x^2|}{1+x^2}=\frac{1}{n}\sup_{0\le x\le b_n}\frac{x(b_n-x)}{1+x^2}\le\frac{b_n}{n}.$$

In view of Theorem 1, the assertion (2.4) does not hold in general for an arbitrary function $f\in C_\rho$, $\rho(x)=1+x^2$. Moreover, the polynomials (1.1) are not able to approximate even the analytic function x^2 on the entire interval $[0,b_n]$ without weight, since (2.7) gives

$$\max_{0\le x\le b_n}[B_n(t^2,x)-x^2]=\frac{b_n^2}{4n},$$

which does not converge to zero as $n\to\infty$.

The affirmative solution of the problem of approximation of function $f(x)=x^2$ on the unbounded interval may be obtained by the consideration of the polynomials of Bernstein-Chlodowsky with $\sqrt{b_n}$ replacing b_n in (1.1) that is the polynomials

$$\tilde{B}_n(f,x)=\sum_{k=0}^{n} f\Big(\frac{k}{n}\sqrt{b_n}\Big)C_n^k\Big(\frac{x}{\sqrt{b_n}}\Big)^k\Big(1-\frac{x}{\sqrt{b_n}}\Big)^{n-k},\quad 0\le x\le\sqrt{b_n}. \tag{2.8}$$

In this case (2.7) gives

$$\tilde{B}_n(t^2,x)=x^2+\frac{x(\sqrt{b_n}-x)}{n},\qquad 0\le x\le\sqrt{b_n}, \tag{2.9}$$

and therefore

$$\max_{0\le x\le\sqrt{b_n}}[\tilde{B}_n(t^2,x)-x^2]=\frac{b_n}{4n},$$

which tends to zero as $n \to \infty$.

We consider now the problem of approximation of arbitrary continuous functions by polynomials (2.8).

Firstly we shall consider the special case.

Lemma 4. *For any continuous function f vanishing on $[a, \sqrt{b_n}]$, where $a > 0$ independent on n,*

$$\lim_{n\to\infty} \sup_{0\le x\le\sqrt{b_n}} |\tilde{B}_n(f,x) - f(x)| = 0\,.$$

P r o o f. Since by the condition, f is bounded, say $|f(x)| \le M$, $0 \le x \le a$, we can write for arbitrary small $\varepsilon > 0$ the inequality

$$|f(\frac{k}{n}\sqrt{b_n}) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(\frac{k}{n}\sqrt{b_n} - x)^2,$$

where $x \in [0, \sqrt{b_n}]$ and $\delta = \delta(\varepsilon)$ independent on n.

By the properties (2.5), (2.6) and (2.9)

$$\sum_{k=0}^{n}(\frac{k}{n}\sqrt{b_n} - x)^2 C_n^k(\frac{x}{\sqrt{b_n}})^k(1 - \frac{x}{\sqrt{b_n}})^{n-k} = \frac{x(\sqrt{b_n} - x)}{n}\,.$$

Therefore

$$\sup_{0\le x\le\sqrt{b_n}} |\tilde{B}_n(f,x) - f(x)| = \varepsilon + \frac{2M}{\delta^2}\frac{b_n}{4n}\,,$$

which gives the proof. ∎

Theorem 5. *Let f be a continuous function on semiaxis $[0,\infty)$, for which*

$$\lim_{x\to\infty} f(x) = k_f < \infty\,.$$

Then

$$\lim_{n\to\infty} \sup_{0\le x\le\sqrt{b_n}} |\tilde{B}_n(f,x) - f(x)| = 0\,.$$

P r o o f. Obviously it is sufficient to prove this theorem in the case of $K_f = 0$. In this case for any $\varepsilon > 0$, there exists a point x_0 such that

$$|f(x)| < \varepsilon, \quad x \ge x_0\,. \tag{2.10}$$

Consider the function g with properties: $g(x) = f(x)$ if $0 \le x \le x_0$, $g(x)$ is linear on $x_0 \le x \le x_0 + \frac{1}{2}$ and $g(x) = 0$ if $x \ge x_0 + \frac{1}{2}$.

Then

$$\sup_{0\le x\le\sqrt{b_n}} |f(x)-g(x)| \le \sup_{x_0\le x\le x_0+\frac{1}{2}} |f(x)-g(x)| + \sup_{x\ge x_0+\frac{1}{2}} |f(x)|$$

and since

$$\max_{x_0\le x\le x_0+\frac{1}{2}} |g(x)| = |f(x_0)|,$$

we have

$$\sup_{0\le x\le\sqrt{b_n}} |f(x)-g(x)| \le 3\varepsilon$$

by the condition (2.10).

Now we obtain

$$\begin{aligned}\sup_{0\le x\le\sqrt{b_n}} |\tilde{B}_n(f,x)-f(x)| &\le \sup_{0\le x\le\sqrt{b_n}} \tilde{B}_n(|f-g|,x)\\ &\quad+ \sup_{0\le x\le\sqrt{b_n}} |\tilde{B}_n(g,x)-g(x)|\\ &\quad+ \sup_{0\le x\le\sqrt{b_n}} |f(x)-g(x)|\\ &\le 6\varepsilon + \sup_{0\le x\le\sqrt{b_n}} |\tilde{B}_n(g,x)-g(x)|,\end{aligned}$$

where $g(x)$ vanishes in $x_0+\frac{1}{2}\le x\le b_n$. By Lemma 4, we obtain the desired result.

It is easy to see that the analog of Theorem 5 holds also for polynomials (1) with condition $\lim_{n\to\infty}\frac{b_n^2}{n}=0$ (replace $\lim_{n\to\infty}\frac{b_n}{n}=0$). Namely we have the following theorem.

Theorem 6. *If the sequence (b_n) in (1) satisfies the condition $\lim_{n\to\infty}\frac{b_n^2}{n}=0$, then for any function f, satisfying the condition of Theorem 5,*

$$\lim_{n\to\infty}\max_{0\le x\le b_n} |B_n(f,x)-f(x)|=0.$$

3. A generalization

We now give a generalization of Bernstein-Chlodowsky polynomials, which can be used to approximate continuous functions on more general weighted spaces.

Let $\omega(x) \geq 1$ be any continuous function for $x \geq 0$.
Let also

$$F_f(t) = f(t)\frac{1+t^2}{\omega(t)}$$

and consider the following generalization of polynomials (1.1)

$$L_n(f,x) = \frac{\omega(x)}{1+x^2}\sum_{k=0}^{n} F_f(\frac{kb_n}{n})C_n^k(\frac{x}{b_n})^k(1-\frac{x}{b_n})^{n-k}, \tag{3.1}$$

where $x \in [0, b_n]$ and b_n has the property as in (1.1). In the case of $\omega(t) = 1+t^2$ the operators (3.1) coincide with (1.1).

Theorem 7. *For a continuous function f, satisfying the condition*

$$\lim_{x\to\infty}\frac{f(x)}{\omega(x)} = K_f < \infty,$$

the equality

$$\lim_{n\to\infty}\sup_{0\leq x\leq b_n}\frac{|L_n(f,x)-f(x)|}{\omega(x)} = 0$$

holds.

Proof. Obviously

$$L_n(f,x) - f(x) = \frac{\omega(x)}{1+x^2}\{\sum_{k=0}^{n} F_f(\frac{kb_n}{n})C_n^k(\frac{x}{b_n})^k(1-\frac{x}{b_n})^{n-k} - F_f(x)\}$$

and therefore

$$\sup_{0\leq x\leq b_n}\frac{|L_n(f,x)-f(x)|}{\omega(x)} = \sup_{0\leq x\leq b_n}\frac{|B_n(F_f,x)-F_f(x)|}{1+x^2}.$$

Also $F_f(x)$ is a continuous function on $[0,\infty)$ satisfying $|F_f(x)| \leq M_f(1+x^2)$, $x \geq 0$, since we have the inequality $|f(x)| \leq M_f\omega(x)$ for f. Therefore, by Proposition 3 we obtain the desired result.

Note that these types of statements also may be obtained for the generalization of Bernstein-Chlodowsky polynomials considered in [5].

Acknowledgement:

The author is thankful to Prof. Dr. A. D. Gadjiev who suggested the problem.

References

[1] G. Bleimann, P. L., Butzer and L. Hahn, A Bernstein-type operator approximating continuous functions on semi-axis, *Indag. Math.* **42**, (1980), 255-262.

[2] F. Chen and Y. Feng, Limit of iterates for Bernstein polynomials defined on a triangle, *Appl. Math. Ser.* **B8**, No 1, (1993), 45-53.

[3] A. D. Gadjiev, The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P. P. Korovkin, *Dokl. Akad. Nauk SSSR*, **218**, No 5 (English Translated in: *Soviet Math. Dokl.* **15** (1974), No 5).

[4] A. D. Gadjiev, R. O. Efendiev, E. İbikli, Generalized Bernstein-Chlodowsky polynomials, *Rocky Mountain Journal of Mathematics*, **28** (1998), No 4.

[5] E. A. Gadjieva and E. İbikli, On generalization of Bernstein - Chlodowsky polynomials, *Hacettepe Bul. of Nat. Sci. and Eng.* **24** (1995), 31-40.

[6] A. Khan Rasul, A note on a Bernstein-Type operator of Blimann, Butzer and Hahn, *Journal of Approximation Theory*, **53** (1988), 295-303.

[7] V. Totik, Uniform approximation by Bernstein-Type operators, *Indag. Math.* **46** (1984), 87-93.

Ankara University, Faculty of Sciences,
Department of Mathematics,
Tandogan, Ankara, TURKEY
E-mail: ibikli@science.ankara.edu.tr

Received 07.12.2002