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ϕ-Variation and Barner-Weil Formula

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The framework of Jorgenson - Lang far-reaching investigations of regularized products and series is enlarged to encompass functions satisfying generalized variation condition at infinity and reciprocal logarithmic growth at zero. Fourier inversion and a general Parseval formula for such functions are proved.

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1. Introduction

The expression

$$(1)\quad W_v(f,\chi) = \lim_{\lambda\to\infty}\left[\int_{-\infty}^{\infty}\frac{e^{\left(\frac{1-|m|}{N_v}-\frac{1}{2}\right)|x|}}{\left|e^{\frac{x}{N_v}}-e^{-\frac{x}{N_v}}\right|}f_\chi(x)\left(1-e^{-\lambda|x|}\right)dx - 2f(0)\log\lambda\right],$$

which appears in A. Weil's "explicit formula" of prime number theory [9], is known as a Weil's functional. (Here χ is a Hecke character belonging to an algebraic number field K with a conductor, v is an archemedean infinite prime spot of K and $m = m(\chi) \in \mathbb{Z}$ is uniquely determined by χ. $N_v = 1$ for real v and $N_v = 2$ for complex v). In a detailed proof of Weil's formula (cf. [7]), to obtain functional W_v makes it necessary to impose restrictions upon the function f near the origin. As K. Barner, who gave a more practical form to the Weil's functional in [3], noted, the advance essentially consists in weakening the conditions imposed on f.

The basic conditions posed on f by Jorgenson and Lang [5], are:

Condition 1. $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$

Condition 2. There exists $\epsilon > 0$ such that $f(x) - f(0) = O(|x|^\epsilon)$ for $x \to 0$.

We will show that Condition 1 can be wakened still further, replacing Jordan variation by ϕ-variation. Condition 2 will be also reconsidered.

2. ϕ–variation and Stieltjes integral

The universal class of this paper is the class W of regulated functions [4, p.145] i.e. functions possessing the one - sided limits at each point. For $f \in W$, we always suppose $2f(x) = f(x+0) + f(x-0)$. If I is an interval with endpoints a and b $(a < b)$, we write $f(I) = f(b) - f(a)$.

Let ϕ be a continuous functions defined on $[0,\infty)$ and strictly increasing from 0 to ∞. A function f is said to be of ϕ bounded variation on I if

$$V_\phi(f,I) = \sup \sum_n \phi(|f(I_n)|) < \infty,$$

where the supremum is taken over all systems $\{I_n\}$ of nonoverlapping subintervals of I.

Example. *$\phi(u) = u$ gives us Jordan variation, and $\phi(u) = u^p$, $p > 1$, corresponds to Wiener variation.*

In the latter case, $V_p(f)$ traditionally denotes the p-th root of $V_\phi(f)$.

The function $\int_0^x \frac{\sin t}{t} dt$ is of bounded ϕ-variation whenever ϕ satisfies conditions $\sum \phi\left(\frac{1}{n}\right) < \infty$ and $\phi(u_1) + \phi(u_2) \leq \phi(u_1 + u_2)$, for $u_1, u_2 > 0$. In particular, the following lemma holds.

Lemma 2.1. *Let $F^A(x) = \int_1^x \frac{\sin At}{t} dt$ for $x > 1$, $F^A = 0$ otherwise. Then $V_p(F^A) = O_p\left(A^{\frac{1}{p}-1}\right)$ $(A \to \infty)$ for every $p > 1$.*

Proof. On $(1,\infty)$ we have $\frac{dF^A}{dx}(x) = \frac{\sin Ax}{x} = 0$ for $Ax = k\pi$, $k > \left[\frac{A}{\pi}\right]$ a natural number. Function F^A is a monotone function on $\left[\frac{k\pi}{A}, \frac{(k+1)\pi}{A}\right]$. Hence

$$Var\left(F^A; \left[\frac{k\pi}{A}, \frac{(k+1)\pi}{A}\right]\right) = \left|F^A\left(\frac{(k+1)\pi}{A}\right) - F^A\left(\frac{k\pi}{A}\right)\right| = \left|\int_{k\pi}^{(k+1)\pi} \frac{\sin t}{t} dt\right| \leq \frac{2}{k\pi} < \frac{1}{k}.$$

Thus $V_p^p\left(F^A; \left[\frac{k\pi}{A}, \frac{(k+1)\pi}{A}\right]\right) \leq \frac{1}{k^p}$ and $V_p^p(F^A) = O\left(\sum_{k=\left[\frac{A}{\pi}\right]}^{\infty} \frac{1}{k^p}\right) = O_p\left(\frac{1}{A^{p-1}}\right)$.

Definition. ([10]) *The Stieltjes integral $\int_a^b f dg$ is said to exist (in a generalized Moore - Pollard sense) with the value J, if given $\epsilon > 0$ there exists a* *finite set of points E in $[a,b]$ such that the sum*

$$\sum_{k=1}^{n} f(\xi_k)\{g(x_k-0)-g(x_{k-1}+0)\}+\sum_{k=1}^{n-1} f(x_k)\{g(x_k+0)-g(x_k-0)\}+$$
$$+f(a)\{g(a+0)-g(a)\}+f(b)\{g(b)-g(b-0)\}$$

differs from J by less than ϵ whenever the points $\{x_k\}$ include all the points of E. $(a=x_0<\xi_1<x_1<\ldots<x_{n-1}<\xi_n<x_n=b)$.

Example. *If f is a step function and g a regulated function, we may take E to be the set of discontinuities of f.*

In case of functions of bounded ϕ-variation, L.C. Young succeeded to prove the following theorems on existence and approximation of this integral.

Theorem 2.A. [10, (5.1.)] *Let f and g be respectively of bounded $\phi-$ and $\psi-$ variation on $[a,b]$ and let ϕ and ψ satisfy the condition*

$$\sum \phi^{-1}\left(\frac{1}{n}\right)\psi^{-1}\left(\frac{1}{n}\right)<\infty .$$

Then the Stieltjes integral of f and g exists and for $a\le\xi\le b$ we have

$$\int_a^b [f(x)-f(\xi)]\,dg(x)\le c\sum \phi^{-1}\left(\frac{A}{n}\right)\psi^{-1}\left(\frac{B}{n}\right),$$

where $A=V_\phi(f,[a,b])$ and $B=V_\psi(g,[a,b])$.

Theorem 2.B. [10, (5.2.)] *Let f, g,ϕ and ψ be as in Theorem 2.A. Then given $\epsilon>0$, there exists $\delta>0$ (independent of f and g) such that*

$$\left|\int_a^b f\,dg-\sum_{k=1}^{n} f(\xi_k)\{g(x_k)-g(x_{k-1})\}\right|<\epsilon,$$

provided that in each $[x_{k-1},x_k]$ determined by $a=x_0\le\xi_1\le x_1\le\ldots\le x_{n-1}\le\xi_n\le x_n=b$, one at least of the functions f and g has its oscillation less than δ.

As a direct consequence of this approximation theorem, we obtain validity of the formula on partial integration.

Theorem 2.1. *Let f and g be respectively of bounded $\phi-$ and $\psi-$ variation on $[a,b]$ and $\sum \phi^{-1}\left(\frac{1}{n}\right)\psi^{-1}\left(\frac{1}{n}\right) < \infty$. If for every $\delta > 0$ there exists a division of $[a,b]$ into subintervals on each of which at least one of the functions f and g has its oscillation less than δ, then*

$$\int_a^b f\,dg = f(b)\,g(b) - f(a)\,g(a) - \int_a^b g\,df.$$

In particular, the formula of partial integration is valid if one of the functions f and g is continuous.

Proof. Given $\epsilon > 0$, according to Theorem 2.B and our assumptions, we may find points $a = x_0 \leq x_1 \leq \ldots \leq x_n = b$ such that

$$\left|\int_a^b f\,dg - \sum_{k=1}^{n} f(x_k)\left(g(x_k) - g(x_{k-1})\right)\right| < \frac{\epsilon}{2} \qquad \text{and}$$

$$\left|\int_a^b g\,df - \sum_{k=1}^{n} g(x_{k-1})\left(f(x_k) - f(x_{k-1})\right)\right| < \frac{\epsilon}{2}$$

(We took x_k, x_{k-1} respectively for ξ_k in Theorem 2.B).

Adding these two inequalities, we end up with

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)\,g(b) - f(a)\,g(a) \qquad \text{(since } \epsilon \text{ was arbitrary).}$$

It is customary to assume that ϕ is a continuous, convex, strictly increasing function on $[0,\infty)$ satisfying conditions

(0_1) $$\lim_{x\to 0} \frac{\phi(x)}{x} = 0$$

(∞_1) $$\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty.$$

These conditions ensure that Young's complementary function of the function ϕ is well defined and that the Young's inequality is satisfied. In what follows we will pose these conditions on ϕ.

3. Harmonic variation and representation of a function by Fourier singular integral

Let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_n} = \infty$. A function f is said to be of Λ-bounded variation on I ($f \in \Lambda BV(I)$) if

$$\sum \frac{|f(I_n)|}{\lambda_n} < \infty$$

for every choice of nonoverlapping intervals $I_n \subset I$. The supremum of these sums is called the $\Lambda-$ variation of f on I and denoted by $V_\Lambda(f, I)$. In case $\Lambda = \mathbb{N}$, one speaks of harmonic bounded variation (HBV).

Perlman has shown that $W(I)$ is precisely the union and $BV(I)$ is the intersection of all $\Lambda BV(I)$ (See [1, p.228] for exact reference, as well as for the following remark).

Remark. $\phi BV(I) \subset HBV(I)$, *if* $\sum \frac{1}{n}\phi^{-1}\left(\frac{1}{n}\right) < \infty$;
$\phi BV(I) \subset \{n^\alpha\} BV(I)$, *if* $\sum \frac{1}{n^\alpha}\phi^{-1}\left(\frac{1}{n}\right) < \infty$, *for* $0 < \alpha < 1$.
For future needs, we note the following, straightforward lemma.

Lemma 3.1. *If* $f \in \Lambda BV(\mathbb{R}) \cap L^1(\mathbb{R})$, *then* $f(x) \to 0$ $(|x| \to \infty)$.

Proof. If we had $\limsup f(x) \neq \liminf f(x)$ $(x \to +\infty$ or $x \to -\infty)$, then the function f could not be of $\Lambda-$bounded variation. The existing $\lim f(x)$ $(x \to +\infty$ or $x \to -\infty)$ has to be 0, otherwise the function would not be integrable on $\mathbb{R}$.

HBV represents a kind of a "natural border" for everywhere pointvise convergence of Fourier series in a sense demonstrated by the following theorem of D. Waterman.

Theorem 3.A. ([8]) a) *If* $f \in HBV[0, 2\pi]$, *then the partial sums of its Fourier series are uniformly bounded. The series converges everywhere and converges uniformly on closed intervals of points of continuity.*

b) *If* $HBV \subset \Lambda BV$ *properly, there is a function* $f \in \Lambda BV[0, 2\pi]$ *whose Fourier series diverges at a point.*

Now, let f be an integrable function over $\mathbb{R}$ and $\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-itx} dx$ its Fourier transform. For such an f and $A > 0$, we define

$$f_A(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(y)\, \frac{\sin A(x-y)}{x-y} dy = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \widehat{f}(t)\, e^{itx} dx.$$

An analogue of Theorem 3.A. is given by

Theorem 3.1. *If $f \in HBV(\mathbb{R}) \cap L^1(\mathbb{R})$, then f_A is bounded independently of A, $f_A(x) \to f(x)$ $(A \to \infty)$ everywhere and convergence is uniform on compact sets of points of continuity of f.*

Proof. We shall start with a simple direct proof of boundness of f_A, uniform in A, as an illustration of a way harmonic variation comes into play in these matters. For a given $\epsilon > 0$ and A, let us choose an odd integer $N = 2L+1$ such that

$$\left| \int_{|t| \geq \frac{N\pi}{A}} f(x+t)\, \frac{\sin At}{t} dt \right| \leq \frac{A}{N\pi} \int_{|t| \geq \frac{N\pi}{A}} |f(x+t)|\, dt \leq \frac{A \|f\|_1}{N\pi} < \epsilon .$$

Notice that

$$\left| \frac{1}{\pi} \int_{|t| \leq \frac{\pi}{A}} f(x+t)\, \frac{\sin At}{t} dt \right| \leq \frac{A}{\pi} \|f\|_\infty \frac{2\pi}{A} = 2\|f\|_\infty$$

$$\begin{aligned} \int_{\frac{\pi}{A}}^{\frac{N\pi}{A}} f(x+t)\, \frac{\sin At}{t} dt &= \int_{\pi}^{N\pi} f\left(x + \frac{t}{A}\right) \frac{\sin t}{t} dt = \sum_{k=1}^{N-1} \int_{k\pi}^{(k+1)\pi} f\left(x + \frac{t}{A}\right) \frac{\sin t}{t} dt \\ &= \int_0^{\pi} \sum_{k=1}^{N-1} f\left(x + \frac{t+k\pi}{A}\right) (-1)^k \frac{\sin t}{t+k\pi} dt . \end{aligned}$$

The integrand is dominated by

$$\sum_{l=1}^{L}\left|f\left(x+\frac{t+(2l-1)\pi}{A}\right)\frac{1}{t+(2l-1)\pi}-f\left(x+\frac{t+2l\pi}{A}\right)\frac{1}{t+2l\pi}\right|$$

$$\leq\sum_{l=1}^{L}\left|f\left(x+\frac{t+(2l-1)\pi}{A}\right)-f\left(x+\frac{t+2l\pi}{A}\right)\right|\frac{1}{t+(2l-1)\pi}$$

$$+\sum_{l=1}^{L}\left|f\left(x+\frac{t+2l\pi}{A}\right)\left(\frac{1}{t+(2l-1)\pi}-\frac{1}{t+2l\pi}\right)\right|\leq V_{II}(f)+\|f\|_{\infty}\sum_{l=1}^{\infty}\frac{1}{(2l-1)^{2}},$$

where $V_{II}(f)=V_{II}(f,\mathbb{R})$. The integral

$$\int_{\frac{-N\pi}{A}}^{\frac{-\pi}{A}} f(x+t)\frac{\sin At}{t}dt$$

may be treated analogously. Hence $\|f_A\|_{\infty}\leq C\left(V_{II}(f)+\|f\|_{\infty}\right)$.

Instead of refining the above argument to obtain the convergence result, that would essentially consist in repeating the argument for Fourier series case, we recall the interplay which exists between these two contexts.

Given $a\in\mathbb{R}$, let ${}^{a}f$ be a periodic function coinciding with f on $(a-\pi,a+\pi)$. Denote by $S_n f$ the $n-th$ partial sum of the Fourier series of ${}^{a}f$. According to [11, Ch. XVII, Theorem (1.3)] $f_A(x)-S_{[A]}f(x)\to 0$ uniformly in each interval totally interior to $(a-\pi,a+\pi)$. Thus, our assertions on convergence follow from Theorem 3.A.

In the next two sections we look for a more precise version of Theorem 3.1 for certain subclasses of $HBV(\mathbb{R})$ relevant to our extension of validity of Barner-Weil formula.

4. Conditions at zero and at infinity

For a regulated function f we will consider the following two conditions:

Condition ϕ. $f\in\phi BV(\mathbb{R})\cap L^{1}(\mathbb{R})$, where $\sum\phi^{-1}\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{\frac{1}{p}}<\infty$ for some $p>1$.

Condition α. There exists $\alpha>1$ such that $f(x)=O\left(|\log|x||^{-\alpha}\right)$ $(x\to 0)$.

Lemma 4.1. *If f satisfies ϕ-condition, then*

$$\|f_A\|_\infty \leq CV_p(F)\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{p}}}\phi^{-1}\left(\frac{B}{n}\right), \text{ where } B = V_\phi(f) = V_\phi(f,\mathbb{R}).$$

Proof. $f_A(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(t)\frac{\sin A(x-t)}{x-t}dt = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(t)\,dF(A(x-t))$, where $F(x) = \frac{1}{\pi}\int\limits_{0}^{x}\frac{\sin t}{t}dt$, as before. Applying Theorem 2.A, for $a < 0 < b$, we obtain

$$\left|\int_a^b [f(t) - f(0)]\,dF(A(x-t))\right| \leq C\sum_{n=1}^{\infty}\phi^{-1}\left(\frac{B_{a,b}}{n}\right)\frac{D_{a,b}}{n^{\frac{1}{p}}},$$

where $B_{a,b} = V_\phi(f,[a,b])$ and $D_{a,b}$ is p-variation of $G(t) = F(A(x-t))$ on $[a,b]$. Obviously $B_{a,b} \leq B = V_\phi(f)$ and $D_{a,b} \leq V_p(F)$. Hence,

$$\begin{aligned}\left|\int_a^b f(t)\frac{\sin A(x-t)}{x-t}dt\right| &\leq |f(0)|\,|F(A(x-b)) - F(A(x-a))| \\ &+ CV_p(F)\sum_{n=1}^{\infty}\phi^{-1}\left(\frac{B}{n}\right)\frac{1}{n^{\frac{1}{p}}}.\end{aligned}$$

Observation that $|F(A(x-b)) - F(A(x-a))| \leq 2\pi$ completes the proof. ∎

Lemma 4.2. *Let g satisfy ϕ-condition. Then, for all $A \geq 1$, we have*

$$\int_1^\infty g(y)\left[\frac{\sin Ay}{y} - \frac{\sin A(x-y)}{x-y}\right]dy = O_g\left(|x|^{\frac{1}{4}\frac{p-1}{p}}\right) \quad (x \to 0)$$

independently of A.

Proof. If g satisfies the assumptions of the theorem, so does $g^-(x) = g(-x)$. Therefore, without loss of generality we may suppose $x > 0$. We will consider two cases.

*C*ase I. $x \leq A^{-4}$

Applying the mean value theorem we have

$$\left|\frac{\sin Ay}{y} - \frac{\sin A(x-y)}{x-y}\right| \ll A^2 x \leq x^{\frac{1}{2}}, \text{ since } x \leq A^{-4}.$$

This gives us

$$\int_1^\infty g(y)\left[\frac{\sin Ay}{y}-\frac{\sin A(x-y)}{x-y}\right]dy \le x^{\frac{1}{2}}\|g\|_1 .$$

Case II. $x > A^{-4}$. Therefore $A^{-1} < x^{\frac{1}{4}}$.

We will consider integrals $\int_1^\infty g(y)\frac{\sin Ay}{y}dy$ and $\int_1^\infty g(y)\frac{\sin A(x-y)}{x-y}dy$ separately.

For an arbitrary $b > 1$ we have $\int_1^b g(y)\frac{\sin Ay}{y}dy = \int_1^b g(y)\,dF^A(y)$. Since the function $F^A(y)$ is continuous, applying the integration by parts formula (Theorem 2.1.) we have

$$\int_1^b g(y)\,dF^A(y) = g(b)\,F^A(b) - \int_1^b F^A(y)\,dg(y).$$

Applying Theorem 2.A. and Lemma 2.1. to the integral on the right side we get:

$$\begin{aligned}\left|\int_1^b F^A(y)\,dg(y)\right| &= \left|\int_1^b \left(F^A(y)-F^A(1)\right)dg(y)\right|\\ &\le K\sum_{n=1}^{\infty}\phi^{-1}\left(\frac{V}{n}\right)\frac{1}{n^{\frac{1}{p}}}A^{\frac{1}{p}-1}, \text{ where } V = V_\phi(g)\,.\end{aligned}$$

Since $g(b)\to 0$ $(b\to\infty)$ and $\left|F^A(x)\right| \le \pi$ for all x, letting $b\to\infty$ we obtain

$$\text{(2)} \qquad \left|\int_1^\infty g(y)\frac{\sin Ay}{y}dy\right| \le CA^{\frac{1}{p}-1} \le Cx^{\frac{1}{4}\frac{p-1}{p}}.$$

Now we will rewrite the second integral in the form:

$$\int_1^\infty g(y)\frac{\sin A(x-y)}{x-y}dy = \int_{1-x}^{1} g(x+y)\frac{\sin Ay}{y}dy + \int_1^\infty g(x+y)\frac{\sin Ay}{y}dy = I_1 + I_2.$$

The function $h(y) = g(x+y)\chi_{(1,+\infty)}(y) \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$. Applying (2) to I_2, we get $|I_2| \le Cx^{\frac{1}{4}\frac{p-1}{p}}$. Assuming that $x \le \frac{1}{2}$, we get the trivial estimate for I_1:

$$|I_1| \le \int_{1-x}^{1} |g(x+y)| \frac{dy}{y} \le 2x \|g\|_\infty .$$

This completes the proof of lemma.

Lemma 4.3. *Let g satisfy ϕ condition and α condition. Then for $A \ge 1$,*

$$\left| \int_0^\infty g(y) \frac{\sin Ay}{y} dy \right| \le C \cdot \max\left\{ (\log A)^{1-\alpha}, A^{-\frac{\alpha}{\alpha+1}\frac{p-1}{p}} \right\}.$$

Proof. Since g satisfies the α condition, there exists $\delta_1 > 0$ such that $|g(y)| \le C \cdot |\log y|^{-\alpha}$, $y \in (0, \delta_1]$. Let $\delta = \min\left\{\delta_1, A^{-\frac{1}{\alpha+1}}\right\}$. We will split the integral into two parts:

$$\int_0^\infty g(y) \frac{\sin Ay}{y} dy = \int_0^\delta g(y) \frac{\sin Ay}{y} dy + \int_\delta^\infty g(y) \frac{\sin Ay}{y} dy = I_1 + I_2.$$

The integral I_1 can be trivially estimated:

$$|I_1| \le C \cdot \int_0^\delta \frac{dy}{|\log y|^\alpha y} \le \frac{C}{\alpha - 1} \cdot |\log \delta|^{-\alpha+1} \le C(g) \cdot (\log A)^{1-\alpha},$$

since $\alpha > 1$ and $\delta \le A^{-\frac{1}{\alpha+1}}$.

The estimate for I_2 is obtained in the same way as in Lemma 4.2, since for $b > \delta$ we have

$$\int_\delta^b g(y) \frac{\sin Ay}{y} dy = \int_1^{\frac{b}{\delta}} g(\delta y)\, dF^{A\delta}(y).$$

Now we easily see that

$$\left| \int_\delta^\infty g(y) \frac{\sin Ay}{y} dy \right| \le C_g (A\delta)^{\frac{1}{p}-1}.$$

If $\delta_1 = \min\left\{\delta_1, A^{-\frac{1}{\alpha+1}}\right\}$, then $|I_2| \le C_g \cdot A^{\frac{1}{p}-1}$, since δ_1 depends only on g. If $A^{-\frac{1}{\alpha+1}} = \min\left\{\delta_1, A^{-\frac{1}{\alpha+1}}\right\}$, then $|I_2| \le C_g \cdot A^{-\frac{\alpha}{\alpha+1}\frac{p-1}{p}}$. In both cases we get $|I_2| \le C_g \cdot A^{-\frac{\alpha}{\alpha+1}\frac{p-1}{p}}$. This completes the proof of the lemma. ∎

Lemma 4.4. *Let g satisfy ϕ condition and α condition. Then for all* $A \ge 1$

$$\left|\int_0^1 g(y)\left[\frac{\sin Ay}{y} - \frac{\sin A(x-y)}{x-y}\right]dy\right| = O_g\left(\left|\log|x|\right|^{1-\alpha}\right) \quad (x \to 0)\,.$$

Proof. We will consider two cases, as in the proof of Lemma 4.2 (we will also assume $x > 0$).

Case I. $x \le A^{-4}$. Then, using the mean value theorem we get

$$\left|\int_0^1 g(y)\left[\frac{\sin Ay}{y} - \frac{\sin A(x-y)}{x-y}\right]dy\right| \le Cx^{\frac{1}{2}}\, \|g\|_1\,.$$

Case II. $x > A^{-4}$, so $A^{-1} < x^{\frac{1}{4}}$ and $A^{-1}x^{-\frac{1}{8}} < x^{\frac{1}{8}}$. Then, $\log A > \frac{1}{4}|\log x|$. Using Lemma 4.3 we obtain

$$\left|\int_0^1 g(y)\,\frac{\sin Ay}{y}dy\right| \le C\cdot\max\left\{|\log x|^{1-\alpha}, x^{\frac{1}{4}\frac{\alpha}{\alpha+1}\frac{p-1}{p}}\right\} = O\left(|\log x|^{1-\alpha}\right),\ x \to 0.$$

It is left to estimate the integral $I = \int_0^1 g(y)\,\frac{\sin A(x-y)}{x-y}dy$. We will write I as the sum of three integrals:

$$I = \int\limits_{|x-y|\le A^{-1}} g(y)\,\frac{\sin A(x-y)}{x-y}dy + \int\limits_{A^{-1}<|x-y|\le A^{-1}x^{-\frac{1}{8}}} g(y)\,\frac{\sin A(x-y)}{x-y}dy$$
$$+ \int\limits_{|x-y|>A^{-1}x^{-\frac{1}{8}}} g(y)\,\frac{\sin A(x-y)}{x-y}dy = I_1 + I_2 + I_3.$$

We will estimate integrals I_1, I_2 and I_3 separately.

In I_1 we will substitute $u = y - x$ to get

$$|I_1| = \left| \int_{-A^{-1}}^{A^{-1}} g(x+u) \frac{\sin Au}{u} du \right| \le A \int_{-A^{-1}}^{A^{-1}} |g(x+u)|\, du$$

$$\le 2 \cdot \max_{-A^{-1} \le u \le A^{-1}} |\log |x+u||^{-\alpha} .$$

Since $|x+u| \le x + A^{-1} \le 2x^{\frac{1}{4}}$ for $x \to 0$, we get

$$|I_1| \le 2 \cdot \left|\log\left(x + A^{-1}\right)\right|^{-\alpha} \le C_1 \cdot |\log x|^{-\alpha} \le C \cdot |\log x|^{1-\alpha} .$$

Using the same substitution as in I_1 we obtain

$$|I_2| = \left| \int_{A^{-1}}^{A^{-1}x^{-\frac{1}{8}}} \left(g(x+u) + g(x-u)\right) \frac{\sin Au}{u} du \right|$$

$$\le C \cdot \left| \int_{A^{-1}}^{A^{-1}x^{-\frac{1}{8}}} \frac{\left(|\log|x+u||^{-\alpha} + |\log|x-u||^{-\alpha}\right)}{u} du \right| .$$

Since $|x+u| \le x + A^{-1}x^{-\frac{1}{8}}$ and $|x-u| \le x + A^{-1}x^{-\frac{1}{8}}$, when $x \to 0$ for $A^{-1} \le u \le A^{-1}x^{-\frac{1}{8}} \le x^{\frac{1}{8}}$, we get $|\log|x \pm u||^{-\alpha} \le C \cdot |\log x|^{-\alpha}$. Now we have

$$|I_2| \le C \cdot |\log x|^{-\alpha} \left| \int_{A^{-1}}^{A^{-1}x^{-\frac{1}{8}}} \frac{du}{u} \right| = C \cdot |\log x|^{-\alpha} \cdot \left|\log x^{-\frac{1}{8}}\right|^{-\alpha} = C \cdot |\log x|^{1-\alpha} .$$

We will write I_3 as the sum of two integrals:

$$\int_{|x-y|>A^{-1}x^{-\frac{1}{8}}} g(y) \frac{\sin A(x-y)}{x-y} dy = \int_{x-y>A^{-1}x^{-\frac{1}{8}}} g(y) \frac{\sin A(x-y)}{x-y} dy$$

$$+ \int_{x-y<-A^{-1}x^{-\frac{1}{8}}} g(y) \frac{\sin A(x-y)}{x-y} dy = J_1 + J_2 .$$

For simplicity we will just look at J_2. J_1 is done in the same way. In J_2 we will change variables, putting $t = y - x$ to get

$$J_2 = \int\limits_{t > A^{-1}x^{-\frac{1}{8}}} g(x+t)\frac{\sin At}{t}dt.$$

We now choose the point x_0 such that $x + A^{-1}x^{-\frac{1}{8}} \leq x_0 \leq x + 2A^{-1}x^{-\frac{1}{8}}$, and decompose J_2 into a sum:

$$J_2 = \int\limits_{t > A^{-1}x^{-\frac{1}{8}}} (g(x+t) - g(x_0))\frac{\sin At}{t}dt + \int\limits_{t > A^{-1}x^{-\frac{1}{8}}} g(x_0)\frac{\sin At}{t}dt = L_1 + L_2.$$

Using the growth condition on g, we get

$$|L_2| \leq C \cdot |\log x_0|^{-\alpha} \cdot \frac{1}{A \cdot A^{-1}x^{-\frac{1}{8}}} \leq C \cdot |\log x|^{1-\alpha},$$

since for $0 < a < b$ we have the trivial estimate

$$\left|\int_a^b \frac{\sin Mx}{x}dx\right| \leq \frac{C}{aM}.$$

Applying Theorem 2.A. and Lemma 2.1. to

$$L_1 = \int\limits_{t > A^{-1}x^{-\frac{1}{8}}} (g(x+t) - g(x_0))\, dF^{x^{-\frac{1}{8}}}(t)$$

we get

$$|L_1| \leq C \cdot x^{\frac{1}{8}\frac{p-1}{p}}.$$

This concludes the estimate of the integral $|J_2|$ and hence the integral I.

5. Fourier inversion formula

The main result of this section is the following uniform version of the Fourier inversion theorem.

Theorem 5.1. *Let g satisfy ϕ condition and α condition. Then for all* $A > 1$

$$g_A(x) - g_A(0) = O_g\left(|\log|x||^{1-\alpha}\right), \quad \textit{when } x \to 0.$$

Proof. We will write

$$\begin{aligned} g_A(x) - g_A(0) &= \frac{1}{\pi}\int_{-\infty}^{\infty} g(y)\left[\frac{\sin A(x-y)}{x-y} - \frac{\sin Ay}{y}\right] dy \\ &= \frac{1}{\pi}\left[\int_0^1 + \int_1^{\infty} + \int_{-\infty}^{-1} + \int_{-1}^{0}\right] = \frac{1}{\pi}\left[I_1 + I_2 + I_3 + I_4\right]. \end{aligned}$$

It is enough to take care of integrals I_1 and I_2, since the other two are estimated in the same way. By Lemma 4.2 and Lemma 4.4 we have $|I_1| + |I_2| \le C \cdot |\log|x||^{1-\alpha}$ what finishes the proof. ∎

The immediate consequence of this theorem is the following assertion that further refines uniformity of convergence, if some stronger growth conditions on g in the neighborhood of zero are assumed.

Theorem 5.2. *Suppose g has M derivatives and $g, g', ..., g^{(M-1)}$ are integrable and bounded when $x \to \pm\infty$. Suppose also that*

$$g^{(M)}(x) = O\left(|\log|x||^{-(M+\alpha)}\right) \quad (x \to 0), \quad \textit{for some } \alpha > 1.$$

Then: *(a) Function g_A has M derivatives and $(g_A)^{(k)} = \left(g^{(k)}\right)_A$, $k = \overline{1, M}$.*
(b) For expansion near the origin, we have

$$g_A(x) - \sum_{k=0}^{M} g_A^{(k)}(0) \cdot \frac{x^k}{k!} = O\left(|x|^M \cdot |\log|x||^{1-\alpha}\right) \quad (x \to 0),$$

where the estimate on the right is independent of A.

Proof. (a) It is enough to prove the statement for $M = 1$. Let $g' \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ and let g be integrable and bounded when $x \to \pm\infty$. Since the function $g(y)\frac{d}{dx}\left(\frac{\sin A(y-x)}{y-x}\right)$ is dominated by $CA^2|g| \in L^1(\mathbb{R})$ we can differentiate under the integral sign in the following integral:

$$\frac{d}{dx}g_A(x) = \frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{\infty} g(y)\frac{\sin A(y-x)}{y-x}dy = \frac{1}{\pi}\int_{-\infty}^{\infty} g(y)\frac{d}{dx}\left(\frac{\sin A(y-x)}{y-x}\right)dy$$
$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} g(y)\,d_y\left(\frac{\sin A(y-x)}{y-x}\right).$$

We may assume that the function $\frac{\sin A(y-x)}{y-x}$ and all its derivatives are continuous, since they can be continuously extended at $y=x$. Applying the classical integration by parts theorem to the continuous function g and function $\frac{\sin A(y-x)}{y-x} \in BV([a,b])$ we get

$$\int_a^b g(y)\,d_y\left(\frac{\sin A(y-x)}{y-x}\right) = g(b)\frac{\sin A(b-x)}{b-x} - g(a)\frac{\sin A(a-x)}{a-x}$$
$$-\int_a^b \frac{\sin A(y-x)}{y-x}dg(y).$$

Since $\frac{\sin A(y-x)}{y-x} \to 0$ when $y \to \pm\infty$ and the function g is bounded at infinity, letting $a \to -\infty$ and $b \to \infty$ we get

$$\frac{d}{dx}g_A(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin A(y-x)}{y-x}dg(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} g'(y)\frac{\sin A(y-x)}{y-x}dy = (g')_A(x).$$

This proves (a) by induction.

(b) Since $g^{(M)}(x) = O\left(|\log|x||^{-(M+\alpha)}\right)$, for $\alpha > 1$, Theorem 5.1. gives us

$$g_A^{(M)}(x) - g_A^{(M)}(0) = O\left(|\log|x||^{-(M+\alpha)+1}\right) \quad (x \to 0).$$

Integrating this estimate from 0 to x, we get

$$g_A^{(M-1)}(x) - g_A^{(M-1)}(0) - x \cdot g_A^{(M)}(0) \le C\int_0^x \frac{dt}{|\log|t||^{(M+\alpha)-1}}$$
$$\le C|x|\int_0^x \frac{dt}{|t|\,|\log|t||^{(M+\alpha)-1}} \le C|x|\,|\log|t||^{-(M+\alpha)+2}.$$

Repeating this argument $(M-1)$ times, we end up with

$$g_A(x) - \sum_{k=0}^{M} g_A^{(k)}(0) \cdot \frac{x^k}{k!} = O\left(|x|^M \cdot |\log|x||^{1-\alpha}\right) \quad (x \to 0).$$

6. A general Parseval formula

It is well-known fact that for any two functions f and g from the Schwartz space S of test functions the elementary Parseval formula $\langle f,g\rangle = \left\langle \widehat{f},\widehat{g}\right\rangle$ holds, where $\langle f,g\rangle$ denotes the usual Hermitian product,

$$\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g}(x)\,dx$$

and $\widehat{f}$ is the Fourier transform of f.

The aim of this section is to prove Parseval formula under less restrictive conditions on f and g. Instead of Hermitian product of f and g we will consider the linear functional $H_{\mu,\varphi}$ defined on S where (μ,φ) is a special pair to be defined below. We will prove that if a regulated function f and its first M derivatives satisfy conditions ϕ and α, then

$$\lim_{A\to\infty} \int_{-A}^{A} \widehat{f}(t)\, H_{\mu,\varphi}(t)\,dt = H_{\mu,\varphi}\left(f^-\right),$$

where $f^-(x) = f(-x)$.

Let $\{p\} = \{p_0, p_1, ...\}$ be a sequence of complex numbers with $\mathrm{Re}(p_0) \le \mathrm{Re}(p_1) \le ...$ increasing to infinity. To every p in this sequence we associate a polynomial B_p and set $b_p(t) = B_p(\log t)$. The asymptotic polynomial at zero is a polynomial $P_q(t) = \sum_{\mathrm{Re}(p)<\mathrm{Re}(q)} b_p(t)\cdot t^p$. The last sum is finite since $\{\mathrm{Re}(p)\}$ is increasing to infinity.

A special pair (μ,φ) consists of:

1° a Borel measure μ on $\mathbb{R}^+$ such that $d\mu(x) = h(x)\,dx$, where h is some bounded measurable function.

2° a measurable function φ on $\mathbb{R}^+$ that has the following two properties:

(a) there exists a complex polynomial at zero $P_0(t)$ such that $\varphi(x) - P_0(x) = O(|\log x|^m)$, $x \downarrow 0$ for some integer $m > 0$;

(b) let M be an integer such that $1- \leq M + \mathrm{Re}(p_0) < 0$. Then both $x^M P_0(x)$ and $\varphi(x)$ are in $L^1(|\mu|)$ outside a neighborhood of zero.

Condition (a) implies

$$\varphi(x)\, e^{itx} - P_0(x)\, e^{itx} = O\left(|\log x|^m\right),\ x \downarrow 0.$$

Since, for a fixed t,

$$P_0(x)\, e^{itx} = \sum_{k=0}^{\infty} P_0(x) \frac{x^k}{k!} (it)^k = \sum_{k=0}^{M} P_0(x) \frac{x^k}{k!} (it)^k + O\left(|x|^{l+1}\right),\ x \downarrow 0,$$

where $l = M + \mathrm{Re}(p_0) \geq -1$, we can write

$$\varphi(x)\, e^{itx} - \sum_{k=0}^{M} u_k(x)\,(it)^k = O\left(|\log x|^m\right),\ x \downarrow 0 \text{ for } u_k(x) = P_0(x)\frac{x^k}{k!}.$$

Given a special pair (μ, φ), a functional $H_{\mu,\varphi}$ on the Schwartz space $\mathcal{S}$ of test functions is defined by

$$H_{\mu,\varphi}(\beta) = \int_0^{\infty} \left(\varphi(x)\,\beta(x) - \sum_{k=0}^{M} (-1)^k u_k(x)\,\beta^{(k)}(0) \right) d\mu(x),\ \beta \in \mathcal{S}.$$

Its "Fourier transform" as a distribution is the function $\widehat{H}_{\mu,\varphi}(t)$ such that

$$\widehat{H}_{\mu,\varphi}(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(\varphi(x)\, e^{-itx} - \sum_{k=0}^{M} u_k(x)\,(-it)^k \right) d\mu(x) = \frac{1}{\sqrt{2\pi}} H_{\mu,\varphi}\left(e^{-itx}\right).$$

For the future needs we will state the following elementary lemma:

Lemma 6.1. *Let f be M times differentiable function. There exist $\beta \in S$ such that $\widehat{\beta}$ has compact support and $f^{(k)}(0) = \beta^{(k)}(0)$, $k = \overline{0,n}$.*

Since $\widehat{\beta}$ has compact support it is obvious that for such a β we have $\beta_A = \beta$ for A large enough.

The following analogue of Parseval's equation generalizes the Barner-Weil formula from [3].

Theorem 6.1. *Let f be M times differentiable function such that $f^{(M)}$ satisfies ϕ condition, and let $f^{(j)} \in HBV(\mathbb{R}) \cap L^1(\mathbb{R})$, for $j \in \{0, ..., M-1\}$.*

Assume that $f^{(M)}(x) - f^{(M)}(0)$ *satisfies condition* α *for some* $\alpha > M+2$. *Then, for special pair* (μ, φ) *we have*

$$\begin{aligned}\lim_{A\to\infty}\int_{-A}^{A}\widehat{f}(t)\,\widehat{H}_{\mu,\varphi}(t)\,dt &= \int_{0}^{\infty}\left(\varphi(x)\,f(-x)-\sum_{k=0}^{M}u_k(x)\,(-1)^k f^{(k)}(0)\right)d\mu(x)\\ &= H_{\mu,\varphi}\left(f^{-}\right).\end{aligned}$$

Proof. Since (μ, φ) is a special pair, the integral

$$\int_{0}^{\infty}\left(\varphi(x)\,e^{-itx}-\sum_{k=0}^{M}u_k(x)\,(-it)^k\right)d\mu(x)$$

is absolutely convergent. Hence, we can apply Fubini's theorem to get

$$\int_{-A}^{A}\widehat{f}(t)\,\widehat{H}_{\mu,\varphi}(t)\,dt = \int_{-A}^{A}\frac{dt}{2\pi}\int_{\mathbb{R}\times\mathbb{R}^+}f(y)\left(\varphi(x)\,e^{-itx}-\sum_{k=0}^{M}u_k(x)\,(-it)^k\right)e^{-ity}dyd\mu(x).$$

Interchanging the order of integration, we obtain on the right-hand side

$$\int_{\mathbb{R}\times\mathbb{R}^+}\frac{1}{\pi}f(y)\left(\varphi(x)\,\frac{1}{2}\int_{-A}^{A}e^{-it(x+y)}dt-\sum_{k=0}^{M}u_k(x)\,\frac{1}{2}\int_{-A}^{A}(-it)^k\,e^{-ity}dt\right)dyd\mu(x).$$

Now, using the differentiation formula

$$\frac{1}{2}\int_{-A}^{A}(-it)^k\,e^{-ity}dt = \left(\frac{d}{dy}\right)^k\left(\frac{\sin Ay}{y}\right)$$

and substituting $-y$ instead of y in the last integral, we get

$$\int_{-A}^{A}\widehat{f}(t)\,\widehat{H}_{\mu,\varphi}(t)\,dt = \tag{3}$$

$$\int_{\mathbb{R}\times\mathbb{R}^+}\frac{f^{-}(y)}{\pi}\left(\varphi(x)\,\frac{\sin A(x-y)}{x-y}-\sum_{k=0}^{M}u_k(x)\left(-\frac{d}{dy}\right)^k\left(\frac{\sin Ay}{y}\right)(-it)^k\right)dyd\mu(x)\ .$$

Let us recall that $f^{(k)}$, $(k=\overline{0,n-1})$ is differentiable and hence continuous function. Since for all $a<0<b$ we have $\left(\frac{d}{dy}\right)^k\left(\frac{\sin Ay}{y}\right)\in BV\left([a,b]\right)$, we can apply integration by parts formula k-times to the integral $\frac{1}{\pi}\int\limits_a^b f^-(y)\left(-\frac{d}{dy}\right)^k\left(\frac{\sin Ay}{y}\right)dy$, in the same way as in the proof of Theorem 5.2. The uniform boundness of the function $\left(\frac{d}{dy}\right)^k\left(\frac{\sin Ay}{y}\right)$ and the fact that $f(x)\to 0$ $(x\to\pm\infty)$ give us that

$$\frac{1}{\pi}\int\limits_{-\infty}^{+\infty} f^-(y)\left(-\frac{d}{dy}\right)^k\left(\frac{\sin Ay}{y}\right)dy=(-1)^k f_A^{(k)}(0).$$

Integrating with respect to y in (3), we have:

$$\text{(4)}\quad \int\limits_{-A}^{A}\widehat{f}(t)\,\widehat{H}_{\mu,\varphi}(t)\,dt=\int\limits_0^{\infty}\left(\varphi(x)\,f_A(-x)-\sum_{k=0}^{M}u_k(x)\,(-1)^k f_A^{(k)}(0)\right)d\mu(x)\,.$$

This finishes the proof of our theorem in the case $f=\beta$, where $\beta\in\mathcal{S}$, $\beta(0)\neq 0$ and function $\widehat{\beta}$ has compact support. Otherwise, for $\delta>0$ we will split the last integral into three parts:

$$\int\limits_0^{\infty}(\varphi(x)-P_0(x))\,f_A(-x)\,d\mu(x)+\int\limits_0^{\delta}P_0(x)\left(f_A(-x)-\sum_{k=0}^{M}(-1)^k\frac{x^k}{k!}f_A^{(k)}(0)\right)d\mu(x)$$
$$+\int\limits_{\delta}^{\infty}P_0(x)\left(f_A(-x)-\sum_{k=0}^{M}(-1)^k\frac{x^k}{k!}f_A^{(k)}(0)\right)d\mu(x)=I_1+I_2+I_3.$$

Lemma 6.1 implies that there exists $\beta\in\mathcal{S}$, $\widehat{\beta}$ with compact support such that $f^{(k)}(0)=\beta^{(k)}(0)$ for $k=\overline{0,M}$. Set $g=f-\beta$. Since the above integrals are linear in f and the statement holds for β, it is enough to prove the theorem for g. It is left to prove that we can apply the dominated convergence theorem and pass to the limit $(A\to\infty)$ under the integral sign in I_1, I_2 and I_3 (with f replaced by g).

Conditions posed on ϕ imply that $\phi BV(I)\subseteq HBV(I)$, so $\left\|g_A^{(k)}\right\|_\infty$ $(k=\overline{0,M})$ is uniformly bounded (Theorem 3.1). Since $(\varphi(x)-P_0(x))\in L^1(\mu)$, we can apply the dominated convergence theorem to I_1.

Let us choose δ such that for $x\in(0,\delta)$ the estimate of the Theorem 5.2 holds for the function g_A. Obviously, δ depends only on g. Theorem 5.2 implies that the integrand in I_2 is bounded by $C\,|x|^{M+\mathrm{Re}(p_0)}\cdot|\log x|^{-\alpha}$, for some $\alpha>1$.

Since $M+\mathrm{Re}(p_0)\geq -1$ the function $h(x)=|x|^{M+\mathrm{Re}(p_0)}\cdot|\log x|^{-\alpha}$ is integrable near zero and we can apply dominated convergence theorem to I_2.

Finally, the uniform boundness of $g_A^{(k)}$ gives us the estimate

$$\left|P_0(x)\left(g_A(-x)-\sum_{k=0}^{M}(-1)^k\frac{x^k}{k!}g_A^{(k)}(0)\right)\right|\leq C|x|^M|P_0(x)|,\ x\geq\delta,$$

for the integral I_3. The condition (b) for a special pair enables us to apply the dominated convergence theorem to I_3.

This completes the proof of the theorem.

Theorem 6.1 and the Gauss formula for the $\Gamma-$function

$$\frac{\Gamma'}{\Gamma}\left(a+i\frac{t}{b}\right)=-\sqrt{2\pi}\widehat{H}_{\mu,\varphi}(t)\,,\ \text{where}$$

$a>0,\ b>0,\ t\in\mathbb{R},\ \mu(x)=e^{-bx},\ \varphi(x)=\dfrac{be^{(1-a)bx}}{1-e^{-bx}}$ and $P_0(x)=\dfrac{1}{x}$, imply the classical Barner - Weil formula for the Weil's functional.

Corollary 6.1. *Let f satisfy the condition ϕ and let $f(x)-f(0)$* satisfy the *condition α, for some $\alpha>2$. Then:*

$$\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\widehat{f}(t)\frac{\Gamma'}{\Gamma}\left(a+i\frac{t}{b}\right)dt=\int_{0}^{\infty}\left(\frac{f(0)}{x}-\frac{be^{(1-a)bx}}{1-e^{-bx}}f(-x)\right)e^{-bx}dx.$$

It is possible to obtain more general corollary with the principal part I_w of a regularized harmonic series instead of $\frac{\Gamma'}{\Gamma}$.

Jorgensen and Lang [5] considered two sequences: sequence $L=\{\lambda_k\}_{k\geq 0}$ of distinct complex numbers with $\lambda_0=0,\ \lambda_k\neq 0,\ (k>0)$ and $A=\{a_k\}_{k\geq 0}$ of complex numbers. To the pair (L,A) they formally associated Dirichlet series $\zeta(s)=\sum\frac{a_k}{\lambda_k^s}$ and theta series $\theta(t)=\sum a_ke^{-\lambda_kt}$, $t>0$.

Their convergence conditions imposed on the Dirichlet series DIR1-DIR3 [5, pp. 129-130] and asymptotic conditions AS1-AS3 [5, pp. 131-132] posed on the function $\theta(t)$ imply that the Laplace-Mellin transform of the θ,

$$\xi(s,z)=LM\theta(s,z)=\int_{0}^{\infty}\theta(t)\,e^{-zt}t^s\frac{dt}{t}$$

is a meromorphic function in s near $s = 1$. The regularized harmonic series, $R(z)$, associated to the pair (L, A) is defined as the constant term of the Laurent expansion of the function $\xi(s, z)$ at $s = 1$.

For any $w \in \mathbb{C}$ with $\mathrm{Re}(w) > 0$ and $\mathrm{Re}(w) > \max_k (-\mathrm{Re}(\lambda_k + z))$ regularized harmonic series can be expressed as the sum

$$R(z+w) = \int_0^\infty (\theta_z(t) - P_0\theta_z(t))\, e^{-wt} dt + S_w(z) = I_w(z) + S_w(z),$$

where $I_w(z)$ is so called "principal part" of $R(z+w)$ and $S_w(z)$ is a polynomial in z of degree less than $-\mathrm{Re}(p_0)$. Here, $\theta_z(t) = e^{-zt}\theta(t)$ and $P_0\theta(t)$ is the principal part of an asymptotic expansion of $\theta_z(t)$ at zero. For $z = a + it$ the principal part of $\theta_z(t)$ can be written as

$$P_0\theta_z(t) = \sum_{k+\mathrm{Re}(p_0)<0} b_p(x) \frac{x^{p+k}}{k!} (-a-it)^k = \sum_{k+\mathrm{Re}(p_0)<0} c_k(a,x)(-it)^k.$$

With this notation, one has

$$I_w(a+it) = \int_0^\infty \left(\theta_a(x) e^{-itx} - \sum_{k+\mathrm{Re}(p_0)<0} c_k(a,x)(-it)^k \right) e^{-wx} dx = \sqrt{2\pi}\widehat{H}_{\mu,\varphi}(t),$$

for $\varphi(x) = \theta_a(x)$; $\mu(x) = e^{-wx}$ and $u_k(x) = c_k(a,x)$.

The asymptotic conditions posed on the $\theta-$function and the condition $\mathrm{Re}(w) > 0$ imply that such a pair (μ, φ) is a special pair.

Our Theorem 6.1 proves that the Jorgenson - Lang general Parseval formula for $I_w(a+it)$ holds for a broader class of test functions, introduced in our paper.

Corollary 6.2. *Let (L, A) be a pair of sequences of complex numbers such that conditions DIR1-DIR3 and AS1-AS3 are satisfied. Let f satisfy conditions of Theorem 6.1 up to an order M for an integer M such that $-1 \leq M + \mathrm{Re}(p_0) < 0$. Then for any $w \in \mathbb{C}$ such that $\mathrm{Re}(w) > 0$ and*

$\mathrm{Re}\,(w) > \max_k \left(-\mathrm{Re}\,(\lambda_k)\right)$ *and* $a \in \mathbb{R}_{\geq 0}$ *we have*

$$\lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \widehat{f}\,(t)\, I_w\,(a+it)\, dt$$

$$= \int_0^\infty \left(\theta_a\,(x)\, f^-\,(x) - \sum_{k+\mathrm{Re}(p_0)<0} c_k\,(a,x)\, f^{(k)}\,(0) \right) e^{-wx} dx.$$

Remark. Theorem 6.1 plays a major role in the proof [2] of a generalization of A. Weil's explicit formula to a fundamental class of functions (introduced by J. Jorgenson and S. Lang in [6]), under less restrictive conditions posed on the test function F.

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