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Measurability of Sets of Pairs of Planes in the Simply Isotropic Space

Adrijan V. Borisov[1], *Margarita G. Spirova*[2]

Presented by P. Kenderov

The measurable sets of pairs of planes and the corresponding invariant densities with respect to the group of the general similitudes and some its subgroups are given.

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1. Introduction

The simply isotropic space $I_3^{(1)}$ is defined [6], [8], [9] as a projective space $\mathbb{P}_3(\mathbb{R})$ in which the absolute consists of a plane ω (*the absolute plane*) and two complex conjugate straight lines f_1, f_2 (*the absolute lines*) into ω with a (real) intersection point F (*the absolute point*). All regular projectivities transforming the absolute figure into itself form the 8-parametric *group* G_8 *of the general simply isotropic similitudes.* Passing on to affine coordinates (x, y, z) a similitude of G_8 can be written in the form [6; p.3]

$$(1)\qquad \begin{aligned} \overline{x} &= a + p(x\cos\varphi - y\sin\varphi), \\ \overline{y} &= b + p(x\sin\varphi + y\cos\varphi), \\ \overline{z} &= c + c_1x + c_2y + c_3z, \end{aligned}$$

where $p > 0, \varphi, a, b, c, c_1, c_2$ and $c_3 \neq 0$ are real parameters.

We shall consider with G_8 and the following its subgroups:

I. $B_7 \subset G_8 \iff p = 1$. It is the group of the simply isotropic similitudes of the δ-distance [6; p.5].

II. $S_7 \subset G_8 \iff c_3 = 1$. It is the group of the simply isotropic similitudes of the s-distance [6; p.6].

III. $W_7 \subset G_8 \iff c_3 = p$. It is the group of the simply isotropic angular similitudes [6; p.18].

IV. $G_7 \subset G_8 \iff \varphi = 0$. It is the group of the boundary simply isotropic similitudes [6; p.8].

V. $V_7 \subset G_8 \iff c_3 p^2 = 1$. It is the group of the volume preserving simply isotropic similitudes [6: p.8].

VI. $G_6 = G_7 \cap V_7$. It is the group of the volume preserving boundary simply isotropic similitudes [6; p.8].

VII. $B_6 = B_7 \cap G_7$. It is the group of the modular boundary motions [6; p.9].

VIII. ${B_6}^{(1)} = B_7 \cap S_7$. It is the group of the simply isotropic motions [6; p.6].

IX. $B_5 = B_6 \cap {B_6}^{(1)}$. It is the group of the unimodular boundary motions [6; p.9].

A plane in ${I_3}^{(1)}$ is said to be *nonisotropic* if its infinite straight line is not incident with the absolute point F; otherwise the plane is said to be *isotropic*.

We underline that much of the common material of the geometry of the simply isotropic space ${I_3}^{(1)}$ can be found in [6], [8] and [9].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [7], G. I. Drinfel'd and A. V. Lucenko [3], [4], [5] we study the measurability of sets of pairs of planes in ${I_3}^{(1)}$ with respect to G_8 and indicated above subgroups. Analogous problems for sets of spheres in ${I_3}^{(1)}$ have been treated in [1].

2. Measurability of a set of pairs of nonisotropic planes

2.1. Measurability of a set of pairs of intersecting nonisotropic planes with respect to G_8

Let (π_1, π_2) be a pair of intersecting nonisotropic planes determined by the equations [6; p.16]

$$\pi_i : z = u_i x + v_i y + w_i, \quad i = 1, 2,$$

where $u_1 v_2 - u_2 v_1 \neq 0$. *The angle* $\psi(\pi_1, \pi_2)$ between π_1 and π_2 is defined by [6; p.17]

$$\psi(\pi_1, \pi_2) = +\sqrt{(u_2 - u_1)^2 + (v_2 - v_1)^2}. \tag{2}$$

The angle (2) is a relative invariant of the group G_8 and an absolute invariant of the group W_7 and $B_6^{(1)}$ [6; p.18]. Under the actoin of (1) the pair

$(\pi_1,\pi_2)(u_1,v_1,w_1,u_2,v_2,w_2)$ is transformed into the pair $(\pi_1',\pi_2')(u_1',v_1',w_1',u_2',v_2',w_2')$ as

$$u_i' = \frac{(u_ic_3+c_1)}{p}\cos\varphi - \frac{(v_ic_3+c_2)}{p}\sin\varphi,$$

$$v_i' = \frac{(u_ic_3+c_1)}{p}\sin\varphi + \frac{(v_ic_3+c_2)}{p}\cos\varphi, \tag{3}$$

$$w_i' = \frac{a(v_ic_3+c_2)-b(u_ic_3+c_1)}{p}\sin\varphi - \frac{a(u_ic_3+c_1)+b(v_ic_3+c_2)}{p}\cos\varphi + w_ic_3 + c,$$

where $i = 1, 2$.

The transformations (3) form the associated group $\overline{G_8}$ of G_8. $\overline{G_8}$ is isomorphic to G_8 and the invariant density with respect to G_8 of the pairs of planes (π_1,π_2), if it exists, coincides with the invariant density with respect to $\overline{G_8}$ of the points $(u_1,v_1,w_1,u_2,v_2,w_2)$ in the set of parameters [7, p.33]. The associated group $\overline{G_8}$ has the infinitesimal operators

$$Y_1 = -u_1\frac{\partial}{\partial w_1} - u_2\frac{\partial}{\partial w_2},\ Y_2 = -v_1\frac{\partial}{\partial w_1} - v_2\frac{\partial}{\partial w_2},\ Y_3 = \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2},$$

$$Y_4 = -u_1\frac{\partial}{\partial u_1} - v_1\frac{\partial}{\partial v_1} - u_2\frac{\partial}{\partial u_2} - v_2\frac{\partial}{\partial v_2},$$

$$Y_5 = -v_1\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial v_1} - v_2\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial v_2}, \tag{4}$$

$$Y_6 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2},\ Y_7 = \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2},$$

$$Y_8 = u_1\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_1} + w_1\frac{\partial}{\partial w_1} + u_2\frac{\partial}{\partial u_2} + v_2\frac{\partial}{\partial v_2} + w_2\frac{\partial}{\partial w_2}.$$

We distinguish the following cases:

(A) $u_1 - u_2 \neq 0$. In this case the infinitesimal operators Y_1, Y_3, Y_4, Y_5, Y_6 and Y_7 are arcwise unconnected and

$$Y_8 = -\frac{w_2 - w_1}{u_2 - u_1}Y_1 + \frac{-u_1w_2 + u_2w_1}{u_2 - u_1}Y_3 - Y_4.$$

Since

$$Y_1\left(-\frac{w_2 - w_1}{u_2 - u_1}\right) + Y_3\left(\frac{-u_1w_2 + u_2w_1}{u_2 - u_1}\right) + Y_4(-1) \neq 0,$$

we deduce that a set of pairs of intersecting nonisotropic planes of type (A) is not measurable and it has not measurable subsets with respect to the group G_8.

(B) $v_2 - v_1 \neq 0$. Now the infinitesimal operators Y_2, Y_3, Y_4, Y_5, Y_6 and Y_7 are arcwise unconnected and

$$Y_8 = -\frac{w_2 - w_1}{v_2 - v_1}Y_2 + \frac{-v_1w_2 + v_2w_1}{v_2 - v_1}Y_3 - Y_4.$$

It is easy to see that

$$Y_2(-\frac{w_2 - w_1}{v_2 - v_1}) + Y_3(\frac{-v_1 w_2 + v_2 w_1}{v_2 - v_1}) + Y_4(-1) \neq 0$$

and therefore a set of pairs of intersecting nonisotropic planes of type (B) is not measurable and it has not measurable subsets with respect to the group G_8.

2.2. Measurability of a set of pairs of parallel nonisotropic planes with respect to G_8

Let (π_1, π_2) be a pair of parallel nonisotropic planes determined by the equations

$$\pi_i : z = ux + vy + w_i, \quad i = 1, 2,$$

where

$$w_2 - w_1 \neq 0. \tag{5}$$

The distance $\alpha(\pi_1, \pi_2)$ between π_1 and π_2 is defined by [6; p.18]

$$\alpha(\pi_1, \pi_2) = w_2 - w_1. \tag{6}$$

The distance (6) is a relative invariant of the group G_8 and an absolute invariant of the group S_7 and $B_6^{(1)}$. Now the corresponding associated group $\overline{G}_8$ of G_8 has the infinitesimal operators

$$\begin{aligned} &Y_1 = -u(\tfrac{\partial}{\partial w_1} + \tfrac{\partial}{\partial w_2}),\ Y_2 = -v(\tfrac{\partial}{\partial w_1} + \tfrac{\partial}{\partial w_2}),\ Y_3 = \tfrac{\partial}{\partial w_1} + \tfrac{\partial}{\partial w_2}, \\ &Y_4 = -u\tfrac{\partial}{\partial u} - v\tfrac{\partial}{\partial v},\ Y_5 = -v\tfrac{\partial}{\partial u} + u\tfrac{\partial}{\partial v},\ Y_6 = \tfrac{\partial}{\partial u},\ Y_7 = \tfrac{\partial}{\partial v}, \\ &Y_8 = u\tfrac{\partial}{\partial u} + v\tfrac{\partial}{\partial v} + w_1\tfrac{\partial}{\partial w_1} + w_2\tfrac{\partial}{\partial w_2}. \end{aligned} \tag{7}$$

From (5) it follows that the infinitesimal operators Y_3, Y_6, Y_7 and Y_8 are arcwise unconnected but $Y_6(-u) + Y_7(-v) \neq 0$ and consequently a set of pairs of parallel nonisotropic planes is not measurable and has not measurable subsets under the group G_8.

We summarise the foregoing results in the following

Theorem 2.2. *Sets of pairs of nonisotropic planes are not measurable wiht respect to the group* G_8 *and have not measurable subsets.*

2.3. Measurability of a set of pairs of intersecting nonisotropic planes with respect to $\mathbf{B_7}$

Consider the set of pairs $\pi_1, \pi_2)$ of intersecting nonisotropic planes. The associated group $\overline{B}_7$ of the group B_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_5, Y_6, Y_7$ and Y_8 from (4) and it acts transitively on the set of parameters $(u_1, v_1, w_1, u_2, v_2, w_2)$. The integral invariant function $f = f(u_1, v_1, w_1, u_2, v_2, w_2)$ satisfies the so-called Deltheil system [2; p.28], [7; p.11]

$$Y_1 f = 0,\ Y_2 f = 0,\ Y_3 f = 0,\ Y_5 f = 0,\ Y_6 f = 0,\ Y_7 f = 0,\ Y_8 f + 6f = 0$$

and has the form

$$f = \frac{h}{[(u_2 - u_1)^2 + (v_2 - v_1)^2]^3},$$

where $h =$ const.

Thus we have the following

Theorem 2.3. *The set of pairs of intersecting nonisotropic planes is measurable with respect to the group B_7 and has the invariant density*

$$d(\pi_1, \pi_2) = \frac{1}{\psi^6}\, du_1 \wedge dv_1 \wedge dw_1 \wedge du_2 \wedge dv_2 \wedge dw_2,$$

where ψ is the angle (2).

2.4. Measurability of a set of pairs of parallel nonsotropic planes with respect to $\mathbf{B_7}$

Now let us consider a set of pairs (π_1, π_2) of parallel nonisotropic planes. The corresponding associated group $\overline{B}_7$ of B_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_5, Y_6, Y_7$ and Y_8 from (7). The group $\overline{B}_7$ acts transitively on the set of parameters (u, v, w_1, w_2). The Deltheil system

$$Y_1 f = 0,\ Y_2 f = 0,\ Y_3 f = 0,\ Y_5 f = 0,\ Y_6 f = 0,\ Y_7 f = 0,\ Y_8 f + 4f = 0$$

has the solution

$$f = \frac{h}{(w_2 - w_1)^4},$$

where $h =$ const.

From here it follows immediately

Theorem 2.4. *The set of pairs of parallel planes is measurable with respect to the group B_7 and has the invariant density*

$$d(\pi_1, \pi_2) = \frac{1}{\alpha^4}\, du \wedge dv \wedge dw_1 \wedge \wedge dw_2,$$

where α is the distance (6).

2.5. Measurability of a set of pairs of nonisotropic planes with respect to $S_7, W_7, G_7, V_7, G_6, B_6, B_6^{(1)}$ and B_5

By arguments similar to the ones used above, we examine the measurability of sets of pairs of nonisotropic planes with respect to all the rest groups. We collect the results in the following table:

group	a set of pairs of intersecting nonisotropic planes	a set of pairs of parallel nonisotropic planes
S_7	$d(\pi_1, \pi_2) = \frac{du_1 \wedge dv_1 \wedge dw_1 \wedge du_2 \wedge dv_2 \wedge dw_2}{\psi^4}$	it is not measurable and has not measurable subsets
W_7	it is not measurable and has not measurable subsets	$d(\pi_1, \pi_2) = \frac{1}{\alpha^2} du \wedge dv \wedge dw_1 \wedge dw_2$
G_7	it is not measurable and has not measurable subsets	it is not measurable and has not measurable subsets
V_7	$d(\pi_1, \pi_2) = \frac{du_1 \wedge dv_1 \wedge dw_1 \wedge du_2 \wedge dv_2 \wedge dw_2}{\psi^{\frac{16}{3}}}$	$d(\pi_1, \pi_2) = \frac{1}{\lvert\alpha\rvert^5} du \wedge dv \wedge dw_1 \wedge dw_2$
G_6	it is not measurable but has the measurable subsets $u_2 - u_1 = h_1(v_2 - v_1), v_2 - v_1 \neq 0,$ $v_2 - v_1 = h_2(u_2 - u_1), u_2 - u_1 \neq 0,$ $h_1, h_2 =$ const with the densities $d(\pi_1, \pi_2) = \frac{du_1 \wedge dv_1 \wedge dw_1 \wedge dv_2 \wedge dw_2}{\lvert v_2 - v_1\rvert^{\frac{13}{3}}},$ $d(\pi_1, \pi_2) = \frac{du_1 \wedge dv_1 \wedge dw_1 \wedge du_2 \wedge dw_2}{\lvert u_2 - u_1\rvert^{\frac{13}{3}}}$	$d(\pi_1, \pi_2) = \frac{1}{\lvert\alpha\rvert^5} du \wedge dv \wedge dw_1 \wedge dw_2$

1	2	3
B_6	it is not measurable but has the measurable subsets $u_2 - u_1 = h_1(v_2 - v_1), v_2 - v_1 \neq 0$, $v_2 - v_1 = h_2(u_2 - u_1), u_2 - u_1 \neq 0$, $h_1, h_2 =$ const with the densities $d(\pi_1, \pi_2) = \frac{du_1 \wedge dv_1 \wedge dw_1 \wedge dv_2 \wedge dw_2}{\lvert w_2 - w_1 \rvert^5}$, $d(\pi_1, \pi_2) = \frac{du_1 \wedge dv_1 \wedge dw_1 \wedge du_2 \wedge dw_2}{\lvert w_2 - w_1 \rvert^5}$	$d(\pi_1, \pi_2) =$ $= \frac{1}{\alpha^4} du \wedge dv \wedge dw_1 \wedge dw_2$
$B_6^{(1)}$	it is not measurable but has the measurable subset $(u_2 - u_1)^2 + (u_2 - u_1)^2 = h^2$, $h =$ const $\neq 0$ with the density $d(\pi_1, \pi_2) = du_1 \wedge dv_1 \wedge dw_1 \wedge dw_2 \wedge dw_2$	it is not measurable but has the measurable subsets $w_2 - w_1 = h, h =$ const $\neq 0$ with the density $d(\pi_1, \pi_2) = du \wedge dv \wedge dw_1$
B_5	it is not measurable, but has the measurable subset $u_2 - u_1 = h_1, v_2 - v_1 = h_2$, $h_1, h_2 =$ const, $\lvert h_1 \rvert + \lvert h_2 \rvert \neq 0$ with the density $d(\pi_1, \pi_2) = du_1 \wedge dv_1 \wedge dw_1 \wedge dw_2$	it is not measurable but has the measurable subset $w_2 - w_1 = h, h =$ const $\neq 0$ with the density $d(\pi_1, \pi_2) = du \wedge dv \wedge dw_1$

3. Measurability of a set of pairs of isotropic planes

3.1. Measurability of a set of pairs of intersecting isotropic planes with respect to G_8

Let (ι_1, ι_2) be a pair of intersecting isotropic planes determined by the equations [6; p.18]

$$\iota_i : u_i x + v_i y + 1 = 0, \quad i = 1, 2,$$

where

$$u_1 v_2 - u_2 v_1 \neq 0. \tag{8}$$

Under the action of (1) the pair $(\iota_1, \iota_2)(u_1, v_1, u_2, v_2)$ is transformed into the pair $(\iota_1', \iota_2')(u_1', v_1', u_2', v_2')$ as

$$(9)\qquad \begin{aligned} u_i' &= \frac{u_i \cos\varphi - v_i \sin\varphi}{(v_i a - u_i b)\sin\varphi - (u_i a + v_i b)\cos\varphi + p}, \\ v_i' &= \frac{u_i \sin\varphi + v_i \cos\varphi}{(v_i a - u_i b)\sin\varphi - (u_i a + v_i b)\cos\varphi + p}. \end{aligned}$$

The transformations (9) form the associated group $\overline{G}_8$, which has the infinitesimal operators

$$(10)\qquad \begin{aligned} Y_1 &= u_1^2 \frac{\partial}{\partial u_1} + u_1 v_1 \frac{\partial}{\partial v_1} + u_2^2 \frac{\partial}{\partial u_2} + u_2 v_2 \frac{\partial}{\partial v_2}, \\ Y_2 &= u_1 v_1 \frac{\partial}{\partial u_1} + v_1^2 \frac{\partial}{\partial v_1} + u_2 v_2 \frac{\partial}{\partial u_2} + v_2^2 \frac{\partial}{\partial v_2}, \\ Y_4 &= -u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1} - u_2 \frac{\partial}{\partial u_2} - v_2 \frac{\partial}{\partial v_2}, \\ Y_5 &= -v_1 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial v_1} - v_2 \frac{\partial}{\partial u_2} + u_2 \frac{\partial}{\partial v_2}. \end{aligned}$$

The group $\overline{G}_8$ acts intransitively on the set of points (u_1, v_1, u_2, v_2) and therefore the set of pairs (ι_1, ι_2) of intersecting isotropic planes has not invariant density under G_8. The system

$$Y_1(f) = 0, \quad Y_2(f) = 0, \quad Y_4(f) = 0, \quad Y_5(f) = 0$$

has an integral

$$f = \frac{u_1 u_2 + v_1 v_2}{u_1 v_2 - v_1 u_2}$$

and it is an absolute invariant of $\overline{G}_8$. Consider the subset of pairs of intersecting isotropic planes satisfying the condition

$$(11)\qquad \frac{u_1 u_2 + v_1 v_2}{u_1 v_2 - v_1 u_2} = h,$$

where $h = \text{const}$. By (8) we have $v_1 \neq h u_1$ or $v_2 \neq h u_2$. We can assume without loss of generality that $v_1 \neq h u_1$. Then the group $\overline{G}_8$ induces on the subset (11) the group G_8^* with the infinitesimal operators

$$(12)\qquad \begin{aligned} Z_1 &= u_1^2 \frac{\partial}{\partial u_1} + u_1 v_1 \frac{\partial}{\partial v_1} + u_2^2 \frac{\partial}{\partial u_2}, \\ Z_2 &= u_1 v_1 \frac{\partial}{\partial u_1} + v_1^2 \frac{\partial}{\partial v_1} + \frac{u_2^2(u_1 + h v_1)}{h u_1 - v_1} \frac{\partial}{\partial u_2}, \\ Z_4 &= -u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1} - u_2 \frac{\partial}{\partial u_2}, \\ Z_5 &= -v_1 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial v_1} - \frac{u_2(u_1 + h v_1)}{h u_1 - v_1} \frac{\partial}{\partial u_2}. \end{aligned}$$

The Deltheil system [2; p.28], [7; p.11]

$$Z_1 f + (3u_1 + 2u_2) f = 0,$$

$$Z_2 f + \left(3v_1 + 2u_2 \frac{u_1 + hv_1}{hu_1 - v_1}\right) f = 0,$$

$$Z_4 f - 3f = 0,$$

$$Z_5 f - \frac{u_1 + hv_1}{hu_1 - v_1} f = 0$$

has only trivial solution $f = 0$. Hence the subset of pairs of the intersecting isotropic planes satisfying (11) is not measurable with respect to the group G_8^*.

So we can state:

Theorem 3.1. *Sets of pairs of intersecting isotropic planes are not measurable with respect to the group G_8 and have not measurable subsets.*

3.2. Measurability of a set of pairs of parallel isotropic planes with respect to G_8

Let (ι_1, ι_2) be a pair of parallel planes determined by the equations

$$\iota_i : x + vy + w_i = 0,$$

where

$$w_2 - w_1 \neq 0.$$

Now the corresponding associated group $\overline{G}_8$ of G_8 has the infinitesimal operators

$$Y_1 = -\frac{\partial}{\partial w_1} - \frac{\partial}{\partial w_2}, \quad Y_2 = vY_1,$$

$$Y_4 = w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2},$$

$$Y_5 = (1 + v^2)\frac{\partial}{\partial v} + vw_1 \frac{\partial}{\partial w_1} + vw_2 \frac{\partial}{\partial w_2}$$

and it acts transitively on the set of parameters (v, w_1, w_2). The integral invariant function $f = f(v, w_1, w_2)$ satisfies the system of R. Deltheil

$$Y_1 f = 0, \quad Y_4 f + 2f = 0, \quad Y_5 f + 4vf = 0$$

and has the form

$$f = \frac{h}{(w_2 - w_1)^2 (1 + v^2)},$$

where $h = \text{const}$.

From these considerations, we have the following

Theorem 3.2. *With respect to the group G_8 a set of pairs (ι_1, ι_2) of parallel isotropic planes is measurable and has the invariant density*

$$d(\iota_1, \iota_2) = \frac{1}{(w_2 - w_1)^2(1 + v^2)} dv \wedge dw_1 \wedge dw_2.$$

3.3. Measurability of a set of pairs of intersecting isotropic planes with respect to B_7

Considering a set of pairs (ι_1, ι_2) of intersecting isotropic planes, we obtain that the associated group $\overline{B}_7$ of the group B_7 has the infinitesimal operators Y_1, Y_2, Y_5 from (10). The group $\overline{B}_7$ acts intransitively on the set of points (u_1, v_1, u_2, v_2). Hence the set of pairs (ι_1, ι_2) has not invariant density under B_7. The system

$$Y_1(f) = 0, \quad Y_2(f) = 0, \quad Y_5(f) = 0$$

has the integral

$$f = \frac{u_1u_2 + v_1v_2}{u_1v_2 - u_2v_1}$$

and it is an absolute invariant of $\overline{B}_7$.

Let us consider the subset of intersecting isotropic planes satisfying the condition

$$h = \frac{u_1u_2 + v_1v_2}{u_1v_2 - u_2v_1}, \tag{13}$$

where $h = \text{const}$. Since $u_1h - v_1 \neq 0$ or $u_2h - v_2 \neq 0$, we can again assume without loss of generality that $u_1h - v_1 \neq 0$. By (13), we have that

$$v_2 = \frac{u_2(v_1h + u_1)}{u_1h - v_1} \tag{14}$$

and the group $\overline{B}_7$ induces on the subset (13) the group B_7^* with the infinitesimal operators Z_1, Z_2, Z_5 from (12), which acts transitively on the set of parameters (u_1, v_1, u_2). By (8) and (14), it is clear that $u_2 \neq 0$. The integral invariant function $f = f(u_1, v_1, u_2)$ satisfies the Deltheil system

$$Z_1(f) = 0, \quad Z_2(f) = 0, \quad Z_5(f) = 0$$

and has the form

$$f = \frac{g(u_1h - v_1)}{u_2^2(u_1^2 + v_1^2)^2},$$

where $g = \text{const}$.

Thus, we can state:

Theorem 3.3. *With respect to the group B_7 a set of pairs of intersecting isotropic planes is not measurable but has the measurable subset*

$$u_1u_2 + v_1v_2 = h(u_1v_2 - u_2v_1), \quad h = const$$

with the density

$$d(\iota_1, \iota_2) = \frac{|u_1h - v_1|}{u_2^2(u_1^2 + v_1^2)^2} du_1 \wedge dv_1 \wedge du_2.$$

3.4. Measurability of a set of pairs of parallel isotropic planes with respect to $\mathbf{B_7}$

By analogous arguments we can also state the following

Theorem 3.4. *With respect to the group B_7 a set of pairs of parallel isotropic planes is not measurable but has the measurable subset*

$$\sqrt{1 + v^2} = h(w_2 - w_1),$$

where $h = const$, with the density

$$d(\iota_1, \iota_2) = \frac{1}{(v^2 + 1)^{\frac{3}{2}}} dv \wedge dw_1.$$

3.5. Measurability of a set of pairs of isotropic planes with respect to $\mathbf{S_7, W_7, G_7, V_7, G_6, B_6, B_6^{(1)}}$ and $\mathbf{B_5}$

Based on arguments similar to the ones used above, we obtain the following results:

group	a set of pairs of intersecting isotropic planes	a set of pairs of parallel isotropic planes
S_7	it is not measurable and has not measurable subsets	$d(\iota_1, \iota_2) = = \frac{1}{(w_2 - w_1)^2(1+v^2)} dv \wedge dw_1 \wedge dw_2$
W_7	it is not measurable and has not measurable subsets	$d(\iota_1, \iota_2) = = \frac{1}{(w_2 - w_1)^2(1+v^2)} dv \wedge dw_1 \wedge dw_2$

1	2	3
G_7	it is not measurable and has not measurable subsets	it is not measurable and but has measurable subset $v = h, \quad h = \text{const}$ with density $d(\iota_1, \iota_2) = \frac{1}{(w_2 - w_1)^2} dw_1 \wedge dw_2$
V_7	it is not measurable and not measurable subsets	$d(\iota_1, \iota_2) =$ $= \frac{1}{(w_2 - w_1)^2 (1 + v^2)} dv \wedge dw_1 \wedge dw_2$
G_6	it is not measurable and has not measurable subsets	it is not measurable but has measurable subset $v = h, \quad h = \text{const}$ with density $d(\iota_1, \iota_2) = \frac{1}{(w_2 - w_1)^2} dw_1 \wedge dw_2$
B_6	it is not measurable but has the measurable subsets $u_1 = h_1 v_1, u_2 = h_2 v_2, \ v_1 \neq 0, v_2 \neq 0,$ $u_1 = h_1 v_1, v_2 = g_2 u_2, \ v_1 \neq 0, u_2 \neq 0,$ $v_1 = g_1 u_1, u_2 = h_2 u_2, \ u_1 \neq 0, v_2 \neq 0,$ $v_1 = g_1 u_1, v_2 = g_2 u_2, \ u_1 \neq 0, u_2 \neq 0,$ $h_1, h_2, g_1, g_2 = \text{const}, h_1 \neq h_2, g_1 \neq g_2,$ $h_1 g_2 \neq 1, h_2 g_1 \neq 1$ with the densities $d(\iota_1, \iota_2) = \frac{dv_1 \wedge dv_2}{v_1^2 v_2^2}, \ d(\iota_1, \iota_2) = \frac{dv_1 \wedge du_2}{v_1^2 u_2^2},$ $d(\iota_1, \iota_2) = \frac{du_1 \wedge dv_2}{u_1^2 v_2^2}, \ d(\iota_1, \iota_2) = \frac{du_1 \wedge du_2}{u_1^2 u_2^2},$	it is not measurable, but has the measurable subsets $w_2 - w_1 = h, \ h = \text{const} \neq 0$ with the densities $d(\iota_1, \iota_2) = dv \wedge dw_1$
$B_6^{(1)}$	it is not measurable but has the measurable subset $u_1 u_2 + v_1 v_2 = h(u_1 v_2 - v_1 u_2),$ $h = \text{const} \neq 0$ with the density $d(\iota_1, \iota_2) = \frac{\lvert u_1 h - v_1 \rvert}{u_2^2 (u_1^2 + v_1^2)} du_1 \wedge dv_1 \wedge du_2$	it is not measurable but has the measurable subsets $\sqrt{1 + v^2} = h(w_2 - w_1),$ $h = \text{const}$ with the density $d(\iota_1, \iota_2) = \frac{1}{(v^2 + 1)^{\frac{3}{2}}} dv \wedge dw_1$

1	2	3
B_5	it is not measurable but has the measurable subsets $u_1 = h_1 v_1, u_2 = h_2 v_2,\ v_1 \neq 0, v_2 \neq 0,$ $u_1 = h_1 v_1, v_2 = g_2 u_2,\ v_1 \neq 0, u_2 \neq 0,$ $v_1 = g_1 u_1, u_2 = h_2 v_2,\ u_1 \neq 0, v_2 \neq 0,$ $v_1 = g_1 u_1, v_2 = g_2 u_2,\ u_1 \neq 0, u_2 \neq 0,$ $h_1, h_2, g_1, g_2 = \text{const}, h_1 \neq h_2, g_1 \neq g_2,$ $h_1 g_2 \neq 1, h_2 g_1 \neq 1$ with the densities $d(\iota_1, \iota_2) = \frac{dv_1 \wedge dv_2}{v_1^2 v_2^2},\ d(\iota_1, \iota_2) = \frac{dv_1 \wedge du_2}{v_1^2 u_2^2},$ $d(\iota_1, \iota_2) = \frac{du_1 \wedge dv_2}{u_1^2 v_2^2},\ d(\iota_1, \iota_2) = \frac{du_1 \wedge du_2}{u_1^2 u_2^2},$	it is not measurable but has the measurable subsets $w_2 - w_1 = h,\ h = \text{const} \neq 0$ with the densities $d(\iota_1, \iota_2) = dv \wedge dw_1$

4. Measurability of a set of pairs of a nonisotropic and an isotropic plane

4.1. Measurability of a set of pairs of a nonisotropic and an isotropic plane with respect to G_8

Let (π_1, ι_2) be a pair of a nonisotropic and an isotropic plane determined by the equations

$$\pi_1: \quad z = u_1 x + v_1 y + w_1,$$
$$\iota_2: \quad u_2 x + v_2 y + 1 = 0,$$

respectively, where

(15) $$u_2^2 + v_2^2 \neq 0.$$

Now the corresponding associated group $\overline{G}_8$ has the infinitesimal operators

$$Y_1 = -u_1 \frac{\partial}{\partial w_1} + u_2^2 \frac{\partial}{\partial u_2} + u_2 v_2 \frac{\partial}{\partial v_2},$$

$$Y_2 = -v_1 \frac{\partial}{\partial w_1} + u_2 v_2 \frac{\partial}{\partial u_2} + v_2^2 \frac{\partial}{\partial v_2}, \quad Y_3 = \frac{\partial}{\partial w_1},$$

$$Y_4 = -u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1} - u_2 \frac{\partial}{\partial u_2} - v_2 \frac{\partial}{\partial v_2},$$

$$Y_5 = -v_1 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial v_1} - v_2 \frac{\partial}{\partial u_2} + u_2 \frac{\partial}{\partial v_2},$$

$$Y_6 = \frac{\partial}{\partial u_1}, \quad Y_7 = \frac{\partial}{\partial v_1}, \quad Y_8 = u_1 Y_6 + v_1 Y_7 + w_1 Y_3.$$

From (15) it follows that the infinitesimal operators Y_3, Y_4, Y_5, Y_6, Y_7 are arcwise unconnected but $Y_6(u_1) + Y_7(v_1) + Y_3(w_1) \neq 0$.

Therefore we can state:

Theorem 4.1. *Sets of pairs of nonisotropic and isotropic planes are not measurable with respect to the group G_8 and have not measurable subsets.*

4.2. Measurability of a set of pairs of nonisotropic and isotropic planes respect to $\mathbf{B_7, S_7, W_7, G_7, V_7, G_6, B_6, B_6^{(1)}}$ and $\mathbf{B_5}$

In manner as above, we establish the following results:

group	a set of pairs of nonisotropic and isotropic planes
B_7	it is not measurable and has not measurable subsets
S_7	it is not measurable and has not measurable subsets
W_7	it is not measurable and has not measurable subsets
G_7	it is not measurable and has not measurable subsets
V_7	it is not measurable and has not measurable subsets
G_6	it is not measurable and has not measurable subsets
B_6	it is not measurable and has not measurable subsets
$B_6^{(1)}$	$\dfrac{1}{(u_2^2+v_2^2)^{\frac{3}{2}}} du_1 \wedge dv_1 \wedge dw_1 \wedge du_2 \wedge dv_2$
B_5	it is not measurable but has measurable subsets $u_2 = hv_2, \quad v_2 \neq 0, \quad h = \text{const},$ $v_2 = gu_2, \quad u_2 \neq 0, \quad g = \text{const},$ with density $d(\pi_1, \iota_2) = \dfrac{1}{v_2^2} du_1 \wedge dv_1 \wedge dw_1 \wedge v_2,$ $d(\pi_1, \iota_2) = \dfrac{1}{u_2^2} du_1 \wedge dv_1 \wedge dw_1 \wedge u_2,$

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[1] *Dept. of Descriptive Geometry*
Univ. of Architecture Civil Eng. and Geodesy
1, Christo Smirnenski Blvd.
Sofia 1421, BULGARIA
e-mail: adribor_fgs@uacg.bg

Received 13.03.2003

[2] *Dept. of Algebra and Geometry*
Shumen University "Ep. K. Preslavski"
115, Alen Mak Str.
Shumen 9712, BULGARIA
e-mail: margspr@fmi.shu-bg.net