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On an Inequality Between the Dyadic Diaphony and the Star-Discrepancy

V. S. Grozdanov, S. S. Stoilova

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The star-discrepancy is a classical measure of the irregularity of the distribution of an arbitrary $s-$dimensional net. Respectively, Zinterhof, Hellekalek and Leeb propose other measures of the irregularity of the distribution, so-called diaphony and dyadic diaphony.

In the present paper the authors prove an inequation, connecting the dyadic diaphony and the star-discrepancy of an arbitrary $s-$dimensional net, $(s \geq 1)$.

This inequation is an analogue of the inequation between the diaphony and star-discrepancy, proved by Fleischer and Stegbuchner.

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1. Introduction

For an arbitrary integer $s \geq 1$, let $E^s = [0,1)^s$ be the s-dimensional unit cube. Let $\omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, be an arbitrary net, composed of N points in E^s. For an arbitrary subinterval $Q = [0, x_1) \times \ldots \times [0, x_s)$ of E^s let $c_Q(\mathrm{x})$, $\mathrm{x} \in E^s$ be the characteristic function on Q. The star-discrepancy $D^*(\omega_N)$ is a classical measure of the irregularity of distribution of the net ω_N and is defined as

$$D^*(\omega_N) = \sup_{Q \subseteq E^s} \left| \frac{1}{N} \sum_{n=1}^{N} c_Q(\mathbf{x}_n) - \int_{E^s} c_Q(\mathbf{x}) d\mathbf{x} \right|.$$

Zinterhof [8] proposes another measure for the irregularity of distribution, called diaphony, which is defined as

$$F(\omega_N) = \left(\sum_{\mathbf{m} \in \mathbf{Z}^s \setminus \{0\}} R^{-2}(\mathbf{m}) \left| \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi \mathrm{i} < \mathbf{m}, \mathbf{x}_n >) \right|^2 \right)^{\frac{1}{2}},$$

where for vector $\mathbf{m} = (m_1, \ldots, m_s)$ with integer coordinates

$$R(\mathbf{m}) = \prod_{i=1}^{s} \max(|m_i|, 1) \quad \text{and} \quad < \mathbf{m}, \mathbf{x}_n >$$

is the inner product of the vectors $\mathbf{m}$ and $\mathbf{x}_n$ $(1 \leq n \leq N)$.

The Rademacher [5] functions of order 2 are defined as:

$$r_0(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and for an arbitrary integer $k \geq 1$ $r_k(x) = r_0(2^k x)$. The Walsh [7] functions of order 2 are defined as: $w_0(x) = 1$, $x \in [0,1)$ and for an arbitrary integer $k \geq 1$ with binary presentation $k = 2^{k_1} + \ldots + 2^{k_m}$, where $k_1 > \ldots > k_m$, $w_k(x) = r_{k_1}(x) \ldots r_{k_m}(x)$, $x \in [0,1)$.

For an vector $\mathbf{k} = (k_1, \ldots, k_s)$ with nonnegative coordinates the $\mathbf{k}-$th Walsh function $w_{\mathbf{k}}(\mathbf{x})$ on E^s is $w_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} w_{k_i}(x_i)$, $\mathbf{x} = (x_1, \ldots, x_s) \in E^s$. Let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $\mathcal{W}(2) = \{w_{\mathbf{k}}\}_{\mathbf{k} \in \mathbf{N}_0^s}$ denote the system of the Walsh functions.

Using the system $\mathcal{W}(2)$, Hellekalek and Leeb [4] introduce a new version of diaphony, the so-called dyadic diaphony, which is defined as

$$F(\mathcal{W}(2); \omega_N) = \left(\frac{1}{3^s - 1} \sum_{\mathbf{k} \neq \mathbf{0}} \rho(\mathbf{k}) \left| \frac{1}{N} \sum_{n=1}^{N} w_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for an arbitrary integer $k \geq 0$

(1) $$\rho(k) = \begin{cases} 2^{-2g}, & \text{if } 2^g \leq k < 2^{g+1}, \quad g \geq 0, \quad g \in \mathbf{Z} \\ 1, & \text{if } k = 0, \end{cases}$$

and for a vector $\mathbf{k} = (k_1, \ldots, k_s)$ with nonnegative coordinates

(2) $$\rho(\mathbf{k}) = \prod_{i=1}^{s} \rho(k_i).$$

The next results, connecting the star-discrepancy $D^*(\omega_N)$ and the diaphony $F(\omega_N)$ of an arbitrary net, are proved:

Stegbuchner [6] gives a $s-$dimensional version of the inequality of LeVeque: For each net ω_N in E^s there exists a positive constant $C_1(s)$, depending on the dimension s, for which

$$(D^*(\omega_N))^{\frac{s+2}{2}} \leq C_1(s) F(\omega_N).$$

Zinterhof and Stegbuchner [9] show an estimation in opposite direction: For each net ω_N in E^s there exists a positive constant $C_2(s)$, depending on the dimension s, such as

(3) $$F(\omega_N) \leq C_2(s) (D^*(\omega_N))^{\frac{1}{2}},$$

Fleischer and Stegbuchner [2] also prove a better variant of the inequality (3): For each net ω_N in E^s there exists a positive constant $C_3(s)$, depending on the dimension s, such as

(4) $$F(\omega_N) \leq C_3(s) D^*(\omega_N).$$

In our paper we will prove an inequality between the dyadic diaphony and the star-discrepancy of arbitrary net in E^s, which is an analogue of the one, given in (4).

We will note some equations which will be useful in our work: For an arbitrary dimension $s \geq 1$ we introduce the sets

$$J_s = \{(s_1, s_2) : 0 \leq s_1, s_2 \leq s, \quad s_1 + s_2 = s\},$$

$$J'_s = \{(s_1, s_2) : 0 \leq s_1 \leq s-1, 0 \leq s_2 \leq s, \quad s_1 + s_2 = s\}$$

and

$$I_s = \{(s_1, s_2, s_3) : 0 \leq s_1, s_2, s_3 \leq s, \quad s_1 + s_2 + s_3 = s\}.$$

Let for $1 \leq j \leq s$ a_j, b_j and t_j be arbitrary real numbers. Then the equations hold

$$\prod_{j=1}^{s} (a_j + b_j) = \sum_{(s_1,s_2)\in J_s} \sum^{*} a_{i_1} \dots a_{i_{s_1}} b_{j_1} \dots b_{j_{s_2}},$$

where the summing in Σ^* realised by $\binom{s}{s_1}$ possible choice of addends from the type $a_{i_1} \dots a_{i_{s_1}} b_{j_1} \dots b_{j_{s_2}}$, $(i_1, \dots, i_{s_1}, j_1, \dots, j_{s_2})$ is a permutation of $(1, 2, \dots, s)$, and

(5) $$\prod_{j=1}^{s} (a_j + b_j + t_j) = \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} a_{i_1} \dots a_{i_{s_1}} b_{j_1} \dots b_{j_{s_2}} t_{m_1} \dots t_{m_{s_3}},$$

where the summing in $\sum^*$ realised by $\binom{s}{s_1}\binom{s-s_1}{s_2}$ possible choice of addends from the type $a_{i_1} \dots a_{i_{s_1}} b_{j_1} \dots b_{j_{s_2}} t_{m_1} \dots t_{m_{s_3}}$, $(i_1, \dots, i_{s_1}, j_1, \dots, j_{s_2}, m_1, \dots, m_{s_3})$ is a permutation of $(1, 2, \dots, s)$. It is obvious that

(6) $$\prod_{j=1}^{s} (a_j + b_j) = a_1 \dots a_s + \sum_{(s_1,s_2)\in J'_s} \sum^{*} a_{i_1} \dots a_{i_{s_1}} b_{j_1} \dots b_{j_{s_2}}.$$

2. Statements of the results

Let an arbitrary real $x \in [0,1)$ and an arbitrary integer $k \geq 0$ have bynary presentations $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ and $k = \sum_{j=0}^{\infty} k_j 2^j$, where $x_j, k_j \in \{0,1\}$. We define $x(0) = 0$, $k(0) = 0$, and for an integer $g \geq 1$ $x(g) = \sum_{j=0}^{g-1} x_j b^{-j-1}$ and $k(g) = \sum_{j=0}^{g-1} k_j 2^j$.

For an arbitrary integer $k \geq 1$ we define the integer $g \geq 0$ by the conditions $2^g \leq k < 2^{g+1}$. If $k = 0$, then we put $g = -1$. For an integer $k \geq 0$ and the corresponding integer $g \geq -1$, satisfying the above condition, and $x \in [0, 1)$ we define the function

$$\varphi(k; g; x) = \begin{cases} 0, & \text{if } k = 0 \\ -w_k(x)(x - x(g+1)) - & \\ -\frac{1}{4}\sqrt{\rho(k)}\left(w_{k(g)}(x(g)) - \delta_{2^g,k}\right), & \text{if } k \neq 0 \end{cases} \tag{7}$$

and $\delta_{i,j}$ is the delta Kronecker symbol.

Let $\omega_N = \{\mathbf{x}_1 = (x_1^{(1)}, \dots, x_1^{(s)}), \dots, \mathbf{x}_N = (x_N^{(1)}, \dots, x_N^{(s)})\}$ be an arbitrary net, composed of N poins in E^s. For an arbitrary vector $\mathbf{k}$ with nonnegative coordinates and for every integer n, $1 \leq n \leq N$ we define the function

$$\Phi(w_{\mathbf{k}}; \omega_N; n) = \sum_{(s_1, s_2) \in J'_s} \frac{1}{4^{s_1}} \sum{}^{*} \prod_{p=1}^{s_1} \sqrt{\rho(k_{i_p})} w_{k_{i_p}}(x_n^{(i_p)}) \prod_{q=1}^{s_2} \varphi(k_{j_q}; g_{j_q}; x_n^{(j_q)}).$$

For an arbitrary vector $\mathbf{k}$ with nonnegative coordinates we define

$$\Phi(w_{\mathbf{k}}; \omega_N) = \frac{1}{N} \sum_{n=1}^{N} \Phi(w_{\mathbf{k}}; \omega_N; n), \tag{8}$$

and

$$S(w_{\mathbf{k}}; \omega_N) = \frac{1}{N} \sum_{n=1}^{N} w_{\mathbf{k}}(\mathbf{x}_n). \tag{9}$$

We put

$$G(\mathcal{W}(2); \omega_N) = \sum_{\mathbf{k} \neq \mathbf{0}} \Phi^2(w_{\mathbf{k}}; \omega_N), \tag{10}$$

and

$$H(\mathcal{W}(2); \omega_N) = \sum_{\mathbf{k} \neq \mathbf{0}} \sqrt{\rho(\mathbf{k})} S(w_{\mathbf{k}}; \omega_N) \Phi(w_{\mathbf{k}}; \omega_N). \tag{11}$$

We will prove the next theorem:

Theorem 1. *For every net ω_N by N points in E^s $(s \geq 1)$ the inequality holds*

$$F(\mathcal{W}(2);\omega_N) \le \frac{12^s}{\sqrt{3^s-1}}\sqrt{(D_N^*(\omega_N))^2 - \frac{1}{9^s}G(\mathcal{W}(2);\omega_N) - \frac{2}{36^s}H(\mathcal{W}(2);\omega_N)}.$$

This result was annonced by the authors in [3]. In this paper we will explain the full proof of Theorem 1.

3. Preliminary notations and results

For $x, y \in [0,1)$ we define the function

$$K(x,y) = c_{[0,y)}(x) - \int_0^1 c_{[0,y)}(x)dy + \frac{3}{4} - y.$$

For the vectors $\mathbf{x} = (x_1, \ldots, x_s)$ and $\mathbf{y} = (y_1, \ldots, y_s)$ in E^s, using the function $K(x,y)$, we define the kernel of the product

$$\mathcal{K}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} K(x_j, y_j).$$

Let $\omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be an arbitrary net of N points in E^s. We define the function

$$\Psi_N(\mathbf{y}) = \frac{1}{N}\sum_{i=1}^{N} \mathcal{K}(\mathbf{x}_i, \mathbf{y}) - \int_{[0,1)^s} \mathcal{K}(\mathbf{x},\mathbf{y})d\mathbf{x}.$$

We have that $\int_{E^s} \mathcal{K}(\mathbf{x},\mathbf{y})d\mathbf{x} = \frac{1}{4^s}$, and for the function $\Psi_N(\mathbf{y})$ we obtain that

$$\Psi_N(\mathbf{y}) = \frac{1}{N}\sum_{i=1}^{N} \mathcal{K}(\mathbf{x}_i, \mathbf{y}) - \frac{1}{4^s}. \tag{12}$$

Lemma 1. *Let $k \ge 1$ be an arbitrary integer and define the integer $g \ge 0$ as $2^g \le k < 2^{g+1}$. For every $x \in [0,1)$ we have the equation*

$$\int_0^x w_k(y)dy = -\frac{1}{4}\sqrt{\rho(k)}w_k(x) + w_k(x)(x - x(g+1)) + \frac{1}{4}\sqrt{\rho(k)}w_{k(g)}(x(g)),$$

where the function $\rho(k)$ is defined by (1).

Proof. For each integer $k \ge 1$, $2^g \le k < 2^{g+1}$ and for every $x \in [0,1)$ we have

$$\int_0^x w_k(y)dy = w_{k(g)}(x(g)) \int_{x(g)}^x r_g(y)dy. \tag{13}$$

Let $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ and $y = \sum_{j=0}^{\infty} y_j 2^{-j-1}$, where $x_j, y_j \in \{0,1\}$ be the binary presentations of x and y.

First, let $x \in \left[x(g), x(g) + \frac{1}{2^{g+1}}\right)$. Then $x_g = 0$, $x(g) = x(g+1)$ and $y_g = 0$. We obtain

$$\int_{x(g)}^{x} r_g(y)dy = (-1)^{x_g}(x - x(g+1)). \tag{14}$$

Second, let $x \in \left[x(g) + \frac{1}{2^{g+1}}, x(g) + \frac{1}{2^g}\right)$. Then $x_g = 1$.

If $y \in \left[x(g), x(g) + \frac{1}{2^{g+1}}\right)$, then $y_g = 0$, and if $y \in \left[x(g) + \frac{1}{2^{g+1}}, x\right)$, then $y_g = 1$. We obtain

$$\int_{x(g)}^{x} r_g(y)dy = \int_{x(g)}^{x(g)+\frac{1}{2^{g+1}}} r_g(y)dy + \int_{x(g)+\frac{1}{2^{g+1}}}^{x} r_g(y)dy$$

$$= \frac{1}{2^{g+1}} + (-1)^{x_g}(x - x(g+1)). \tag{15}$$

The formulae (14) and (15) we write in the form

$$\int_{x(g)}^{x} r_g(y)dy = \frac{1-(-1)^{x_g}}{2^{g+2}} + (-1)^{x_g}(x - x(g+1)). \tag{16}$$

From (13) and (16) we obtain

$$\begin{aligned}\int_0^x w_k(y)dy &= -\frac{1}{4} w_{k(g)}(x(g))(-1)^{x_g}\frac{1}{2^g} + \frac{1}{4} w_{k(g)}(x(g))\frac{1}{2^g} \\ &\quad + w_{k(g)}(x(g))(-1)^{x_g}(x - x(g+1)) \\ &= -\frac{1}{4}\sqrt{\rho(k)}w_k(x) + w_k(x)(x - x(g+1)) + \frac{1}{4}\sqrt{\rho(k)}w_{k(g)}(x(g)),\end{aligned}$$

so, Lemma 1 is proved. ◻

Lemma 2. *For an arbitrary integer $k \geq 0$ and $x \in [0,1)$ we have the equation*

$$\int_0^1 K(x,y)w_k(y)dy = \frac{1}{4}\sqrt{\rho(k)}w_k(x) + \varphi(k;g;x),$$

where the functions $\rho(k)$ and $\varphi(k;g;x)$ are defined by the equations (1) and (7).

P r o o f. Let $k=0$. Then from the equations $w_0(x)=1$, $\int_0^1 K(x,y)w_0(y)dy=\frac{1}{4}$, $\rho(0)=1$, and for $x\in[0,1)$ $\varphi(0;-1;x)=0$ we obtain

$$\int_0^1 K(x,y)w_0(y)dy=\frac{1}{4}\sqrt{\rho(0)}w_0(x)+\varphi(0;-1;x)=\frac{1}{4}\sqrt{\rho(k)}w_k(x)+\varphi(k;g;x).$$

If $k\neq 0$, then we have the equations

$$\int_0^1 K(x,y)w_k(y)dy=\int_0^1 c_{[0,y)}(x)w_k(y)dy-\int_0^1 yw_k(y)dy. \tag{17}$$

The equations hold

$$\int_0^1 c_{[0,y)}(x)w_k(y)dy=\int_x^1 w_k(y)dy=\int_0^1 w_k(y)dy-\int_0^x w_k(y)dy$$

$$=-\int_0^x w_k(y)dy. \tag{18}$$

Fine [1] proved that for every integer $k\geq 1$

$$\int_0^1 yw_k(y)dy=-2^{-(g+2)}\delta_{2^g,k}. \tag{19}$$

From (1), (7), (17), (18), (19) and Lemma 1 we obtain

$$\int_0^1 K(x,y)w_k(y)dy$$

$$=\frac{1}{4}\sqrt{\rho(k)}w_k(x)-w_k(x)(x-x(g+1))-\frac{1}{4}\sqrt{\rho(k)}w_{k(g)}(x(g))+2^{-(g+2)}\delta_{2^g,k}$$

$$=\frac{1}{4}\sqrt{\rho(k)}w_k(x)+\varphi(k;g;x),$$

so, Lemma 2 is proved. ∎

Lemma 3. *Let $\omega_N=\{\mathbf{x}_1=(x_1^{(1)},\dots,x_1^{(s)}),\dots,\mathbf{x}_N=(x_N^{(1)},\dots,x_N^{(s)})\}$ be an arbitrary net in E^s. The* **k**−*th Fourier-Walsh coefficient of the function $\Psi_N(\mathbf{y})$, defined by (12) satisfies the equation*

$$\widehat{\Psi}_N(\mathbf{k})=\begin{cases}0, & \text{if } \mathbf{k}=\mathbf{0}\\ \frac{1}{4^s}\sqrt{\rho(\mathbf{k})}S(w_{\mathbf{k}};\omega_N)+\Phi(w_{\mathbf{k}};\omega_N), & \text{if } \mathbf{k}\neq\mathbf{0},\end{cases}$$

where the functions $\rho(\mathbf{k})$, $S(w_{\mathbf{k}};\omega_N)$ and $\Phi(w_{\mathbf{k}};\omega_N)$ are defined respectively by the equations (2), (9) and (8).

P r o o f. For an arbitrary vector $\mathbf{k}$ the Fourier-Walsh coefficient of the function $\Psi_N(\mathbf{y})$ is

$$\widehat{\Psi}_N(\mathbf{k}) = \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{s}\int_0^1 K(x_i^{(j)},y_j)w_{k_j}(y_j)dy_j - \frac{1}{4^s}\prod_{j=1}^{s}\int_0^1 w_{k_j}(y_j)dy_j. \tag{20}$$

Let $\mathbf{k}=\mathbf{0}$. From Lemma 2, when $k=0$ and (20) we obtain $\widehat{\Psi}_N(\mathbf{0})=0$.

Let now $\mathbf{k}\neq\mathbf{0}$. Then there exists some j, $1\leq j\leq s$, that $k_j\neq 0$. Then from (6), (20) and Lemma 2 we obtain

$$\widehat{\Psi}_N(\mathbf{k}) = \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{s}\int_0^1 K(x_i^{(j)},y_j)w_{k_j}(y_j)dy_j$$

$$= \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{s}\left[\frac{1}{4}\sqrt{\rho(k_j)}w_{k_j}(x_i^{(j)}) + \varphi(k_j;g_j;x_i^{(j)})\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{s}\frac{1}{4}\sqrt{\rho(k_j)}w_{k_j}(x_i^{(j)})$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{(s_1,s_2)\in J_s'}\sum\nolimits^{*}\left[\prod_{p=1}^{s_1}\frac{1}{4}\sqrt{\rho(k_{i_p})}w_{k_{i_p}}(x_i^{(i_p)})\right]\left[\prod_{q=1}^{s_2}\varphi(k_{j_q};g_{j_q};x_i^{(j_q)})\right]$$

$$= \frac{1}{4^s}\sqrt{\rho(\mathbf{k})}\frac{1}{N}\sum_{i=1}^{N}w_{\mathbf{k}}(\mathbf{x}_i)$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{(s_1,s_2)\in J_s'}\sum\nolimits^{*}\left[\prod_{p=1}^{s_1}\frac{1}{4}\sqrt{\rho(k_{i_p})}w_{k_{i_p}}(x_i^{(i_p)})\right]\left[\prod_{q=1}^{s_2}\varphi(k_{j_q};g_{j_q};x_i^{(j_q)})\right].$$

Finally, from the significations (9) and (8) we obtain

$$\widehat{\Psi}_N(\mathbf{k}) = \frac{1}{4^s}\sqrt{\rho(\mathbf{k})}S(w_{\mathbf{k}};\xi_N) + \Phi(w_{\mathbf{k}};\xi_N),$$

so Lemma 3 is proved. ■

Lemma 4. *Let ω_N be an arbitrary net of N points in E^s. The equation holds*

$$\|\Psi_N\|_{L_2}^2 = \frac{3^s-1}{4^{2s}}F^2(\mathcal{W}(2);\omega_N) + \frac{2}{4^s}H(\mathcal{W}(2);\omega_N) + G(\mathcal{W}(2);\omega_N),$$

where the functions $G(\mathcal{W}(2);\omega_N)$ and $H(\mathcal{W}(2);\omega_N)$ are defined by the equations (10) and (11).

Proof. By the Parseval formula we have $\|\Psi_N\|_{L_2}^2 = \sum_{\mathbf{k}\in \mathbb{N}_0{}^s} |\widehat{\Psi}_N(\mathbf{k})|^2$. From Lemma 3 we have that $\widehat{\Psi}_N(\mathbf{0}) = 0$ and obtain

$$\|\Psi_N\|_{L_2}^2 = \sum_{\mathbf{k}\neq\mathbf{0}} \left[\frac{1}{4^s}\sqrt{\rho(\mathbf{k})}S(w_{\mathbf{k}};\omega_N) + \Phi(w_{\mathbf{k}};\omega_N)\right]^2$$

$$= \frac{1}{4^{2s}}\sum_{\mathbf{k}\neq\mathbf{0}} \rho(\mathbf{k})|S(w_{\mathbf{k}};\omega_N)|^2 + \sum_{\mathbf{k}\neq\mathbf{0}} \Phi^2(w_{\mathbf{k}};\omega_N)$$

$$+\frac{2}{4^s}\sum_{\mathbf{k}\neq\mathbf{0}} \sqrt{\rho(\mathbf{k})}S(w_{\mathbf{k}};\omega_N)\Phi(w_{\mathbf{k}};\omega_N)$$

$$= \frac{3^s-1}{4^{2s}}F^2(\mathcal{W}(2);\omega_N) + \frac{2}{4^s}H(\mathcal{W}(2);\omega_N) + G(\mathcal{W}(2);\omega_N),$$

so, Lemma 4 is proved. ∎

Lemma 5 *Let ω_N be an arbitrary net of N points in E^s. The following equation holds*

$$\|\Psi_N\|_\infty < 3^s D^*(\omega_N).$$

Proof. For arbitrary vectors $\mathbf{x} = (x_1,\ldots,x_s)$ and $\mathbf{y} = (y_1,\ldots,y_s)$ in E^s we introduce the significations $a_j = c_{[0,y_j)}(x_j)$, $b_j = -\int_0^1 c_{[0,y_j)}(x_j)dy_j$, and $t_j = \frac{3}{4} - y_j$. According to (5) for $\mathcal{K}(\mathbf{x},\mathbf{y})$ we obtain

$$\mathcal{K}(\mathbf{x},\mathbf{y}) = \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*}(-1)^{s_2}t_{m_1}\ldots t_{m_{s_3}} c_{[0,y_{i_1})}(x_{i_1})\ldots c_{[0,y_{i_{s_1}})}(x_{i_{s_1}})$$

$$\times\int_0^1\cdots\int_0^1 c_{[0,y_{j_1})}(x_{j_1})\ldots c_{[0,y_{j_{s_2}})}(x_{j_{s_2}})dy_{j_1}\ldots dy_{j_{s_2}}$$

$$= \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*}(-1)^{s_2}t_{m_1}\ldots t_{m_{s_3}} \tag{21}$$

$$\times\int_0^1\cdots\int_0^1 c_{[0,y_{i_1})}(x_{i_1})\ldots c_{[0,y_{i_{s_1}})}(x_{i_{s_1}})c_{[0,y_{j_1})}(x_{j_1})\ldots c_{[0,y_{j_{s_2}})}(x_{j_{s_2}})dy_{j_1}\ldots dy_{j_{s_2}}.$$

We signify $Q = [0,z_1)\times\ldots\times[0,z_s)$, where $z_j = y_j$ for $j\in\{i_1,\ldots,i_{s_1},j_1,\ldots,j_{s_2}\}$ and $z_j = 1$ for $j\in\{m_1,\ldots,m_{s_3}\}$. Then from (21) and (12) we obtain

$$\Psi_N(\mathbf{y}) = \frac{1}{N}\sum_{i=1}^{N} \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} (-1)^{s_2} t_{m_1} \dots t_{m_{s_3}}$$

$$\times \int_0^1 \dots \int_0^1 c_Q(\mathbf{x}_i) dy_{j_1} \dots dy_{j_{s_2}} - \frac{1}{4^s}$$

$$= \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} (-1)^{s_2} t_{m_1} \dots t_{m_{s_3}} \int_0^1 \dots \int_0^1 \left[\frac{1}{N}\sum_{i=1}^{N} c_Q(\mathbf{x}_i)\right] dy_{j_1} \dots dy_{j_{s_2}} - \frac{1}{4^s}$$

$$= \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} (-1)^{s_2} t_{m_1} \dots t_{m_{s_3}}$$

$$\times \int_0^1 \dots \int_0^1 \left[\frac{1}{N}\sum_{i=1}^{N} c_Q(\mathbf{x}_i) - V(Q)\right] dy_{j_1} \dots dy_{j_{s_2}} \tag{22}$$

$$+ \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} (-1)^{s_2} t_{m_1} \dots t_{m_{s_3}} \int_0^1 \dots \int_0^1 V(Q) dy_{j_1} \dots dy_{j_{s_2}} - \frac{1}{4^s}.$$

We have the equations

$$\int_0^1 \dots \int_0^1 V(Q) dy_{j_1} \dots dy_{j_{s_2}} = y_{i_1} \dots y_{i_{s_1}} \int_0^1 y_{j_1} dy_{j_1} \dots \int_0^1 y_{j_{s_2}} dy_{j_{s_2}}$$

$$= y_{i_1} \dots y_{i_{s_1}} \left(\frac{1}{2}\right)^{s_2},$$

$$\sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} (-1)^{s_2} t_{m_1} \dots t_{m_{s_3}} \int_0^1 \dots \int_0^1 V(Q) dy_{j_1} \dots dy_{j_{s_2}}$$

$$= \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} \left(-\frac{1}{2}\right)^{s_2} y_{i_1} \dots y_{i_{s_1}} t_{m_1} \dots t_{m_{s_3}}$$

$$= \prod_{j=1}^{s} \left(y_j - \frac{1}{2} + t_j\right) = \frac{1}{4^s}. \tag{23}$$

From (22) and (23) we obtain

$$\Psi_N(\mathbf{y}) = \sum_{(s_1,s_2,s_3)\in I_s} \sum^{*} (-1)^{s_2} t_{m_1} \dots t_{m_{s_3}}$$

$$\times \int_0^1 \cdots \int_0^1 \left[\frac{1}{N} \sum_{i=1}^{N} c_Q(\mathbf{x}_i) - V(Q) \right] dy_{j_1} \ldots dy_{j_{s_2}}.$$

Then

$$\|\Psi_N\|_\infty$$

$$= \sup_{\mathbf{y} \in E^s} \left| \sum_{(s_1,s_2,s_3) \in I_s} \sum^{*} (-1)^{s_2} t_{m_1} \ldots t_{m_{s_3}} \right.$$

$$\left. \times \int_0^1 \cdots \int_0^1 \left[\frac{1}{N} \sum_{i=1}^{N} c_Q(\mathbf{x}_i) - V(Q) \right] dy_{j_1} \ldots dy_{j_{s_2}} \right|$$

$$\leq \sum_{(s_1,s_2,s_3) \in I_s} \sum^{*} |t_{m_1}| \ldots |t_{m_{s_3}}|$$

$$\times \int_0^1 \cdots \int_0^1 \sup_{\mathbf{y} \in E^s} \left| \frac{1}{N} \sum_{i=1}^{N} c_Q(\mathbf{x}_i) - V(Q) \right| dy_{j_1} \ldots dy_{j_{s_2}}. \tag{24}$$

We have that

$$\sup_{\mathbf{y} \in [0,1)^s} \left| \frac{1}{N} \sum_{i=1}^{N} c_Q(\mathbf{x}_i) - V(Q) \right|$$

$$\leq \sup_{\mathbf{y} \in [0,1)^s} \sup_{Q \subseteq [0,1)^s} \left| \frac{1}{N} \sum_{i=1}^{N} c_Q(\mathbf{x}_i) - V(Q) \right| = D^*(\omega_N). \tag{25}$$

From (24), (25) and (5) we obtain that

$$\|\Psi_N\|_\infty \leq D^*(\omega_N) \sum_{(s_1,s_2,s_3) \in I_s} \sum^{*} |t_{m_1}| \ldots |t_{m_{s_3}}|$$

$$\leq D^*(\omega_N) \sum_{(s_1,s_2,s_3) \in I_s} \sum^{*} \left(\frac{3}{4}\right)^{s_3} = \left(\frac{11}{4}\right)^{s} D^*(\omega_N),$$

finally $\|\Psi_N\|_\infty < 3^s D^*(\omega_N)$ and Lemma 5 is proved.

4. A proof of Theorem 1

From Lemmas 4 and 5 we obtain

$$\frac{3^s - 1}{4^{2s}} F^2(\mathcal{W}(2); \omega_N) = \|\Psi_N\|^2_{L_2} - \frac{2}{4^s} H(\mathcal{W}(2); \omega_N) - G(\mathcal{W}(2); \omega_N)$$

$$\leq \|\Psi_N\|_\infty^2 - \frac{2}{4^s}H(\mathcal{W}(2);\omega_N) - G(\mathcal{W}(2);\omega_N)$$

$$< 3^{2s}(D^*(\omega_N))^2 - \frac{2}{4^s}H(\mathcal{W}(2);\omega_N) - G(\mathcal{W}(2);\omega_N).$$

From the last inequation we obtain

$$F(\mathcal{W}(2);\omega_N) < \frac{12^s}{\sqrt{3^s-1}}\sqrt{(D^*(\omega_N))^2 - \frac{1}{9^s}G(\mathcal{W}(2);\omega_N) - \frac{2}{36^s}H(\mathcal{W}(2);\omega_N)},$$

and Theorem 1 is proved.

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Department of Mathematics
South West University
66 Ivan Mihailov str.
2700 Blagoevgrad, BULGARIA
e-mail: vassgrozdanov@yahoo.com
e-mail: stanislavast@yahoo.com

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