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Reconstruction of Functions from their Radon Projections

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We describe a numerical method for approximate reconstruction of a function on the unit disk on the basis of its Radon transform $P_f(t, \theta)$ given for a finite number of the rotation angle θ.

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1. Introduction

The natural approach to approximation of functions in the univariate case is based on information about f at a finite number of points. This is because a table of function values is a standard type of information which appears in practical problems and processes. In addition, the simplicity and universality of Lagrange interpolation formulas, Gauss quadrature formula, other classical results in univariate approximation confirms such a choice of the information data. The direct transformation of the univariate results to the multivariate case faces various difficulties. For instance, point-wise interpolation on the plane is no more solvable for arbitrary choice of the nodes. The recent development in tomography, as well as the power of Radon transform and other results in multivariate interpolation suggest as a reasonable choice of the data when working with bivariate functions, the values of linear integrals over segments. Many classical approximation problems admit natural extension in the multivariate case (in $\mathbb{R}^d$, if the approximation is based on integrals over $(d-1)-$ dimensional hyperplanes.) The nice multivariate extension of Lagrange interpolation formula obtained recently by Hakopian [5] is an example.

In this paper we develop numerical methods for reconstruction of bivariate functions on the unit disk $B := \{(x,y) : x^2 + y^2 \leq 1\}$ on the basis of linear integrals $\int_I f\,ds$ over a finite number of chords I. Our study relies on the fact that every function f from $L_2(B)$ is completely determined by its Radon transform

$$P_f(t,\xi) := \int_{\xi^\perp} f(t\xi + s)\,ds\,, \quad -1 \leq t \leq 1\,,$$

where ξ is an element of the unit sphere S and ξ^p is the hyperplane through the origin which is perpendicular to ξ.

In Section 3 we reconstruct on the unit disk a polynomial $f \in \Pi_n(\mathbb{R}^2)$ from its Radon projections $\{P_f(t,\theta_i)\}_{i=1}^n$ for a set of $(n+1)$ angles θ_i. Moreover, every $f \in L_2(B)$ can be reconstructed by the corresponding Tchebiceff- Fourier series. For $f \in C(B)$, the rate of convergence of the approximating polynomial to f is estimated. In Section 4 we present a method for reconstruction of a polynomial f from its projections over all chords starting from a point $M_i = (\cos\alpha_i, \sin\alpha_i)$ on the unit circle, for $i = 0,\ldots,n$, $0 \leq \alpha_0 < \ldots < \alpha_n < 2\pi$. The method is based on Hakopian's interpolation formula [2]. Namely, f is searched as a linear combination of some ridge polynomials $P_k(x\cos\theta_k + y\sin\theta_k)$, each P_k is associated to a chord I_i, such that the integral of P_k over the chord I_i is δ_{ik}.

Results from computer experiments are attached.

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2. Preliminaries

A main topic in our studies is that of ridge functions. Let $\theta \in [0, 2\pi)$ and $\rho(t)$ be a real valued function.

Definition 2.1. *The function $\rho(\theta,x,y) := \rho(x\cos\theta + y\sin\theta)$ is called a ridge function corresponding to ρ, with direction θ.*

Obviously the ridge function $\rho(\theta,x,y)$ is constant along any line of direction θ. We denote by $\Pi_n\,(\mathbb{R}^2)$ the set of all algebraic polynomials of total degree n, i. e. ,

$$\Pi_n\left(\mathbb{R}^2\right) := \left\{\sum_{i+j\leq n} a_{ij}x^iy^j\right\}.$$

In what follows we consider functions $f(x,y)$ which are integrable on the unit disk $B := \{(x,y) :\ x^2 + y^2 \leq 1\}$. Without loss of generality we can suppose that $f(x,y)$ is supported on B.

Given t and $\theta \in [0, \pi)$, we define the line

$$I(t,\theta) = \{(x,y) : \; x\cos\theta + y\sin\theta = t\}.$$

We assume that the intersection of $I(t,\theta)$ and B is not empty.

The formula

$$P_f(t,\theta) = \int_{-\infty}^{+\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds$$

defines the projection $P_f(t,\theta)$ of f along the line $I(t,\theta)$. Since $f(x,y) = 0$ outside B, the projection can be reduced to

$$P_f(t,\theta) = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds.$$

Let us mention the following simple but useful property of the projection:

$$P_f(t,\theta+\pi) = P_f(-t,\theta).$$

That is why we can assume that $0 \le \theta < \pi$ when it is more convenient. By a fundamental result, due to Radon [9], every function $f \in L_2(B)$ is uniquely determined by its Radon transform:

$$f \longrightarrow \{P_f(t,\theta),\; 0 \le \theta < \pi\}.$$

Theorema 2.1. *Every algebraic polynomial $P \in \Pi_n(\mathbb{R}^2)$ can be represented as a sum of $n+1$ ridge polynomials. More precisely for any preassigned $\theta_0, \ldots, \theta_n, \theta_i \neq \theta_j, i \neq j, \theta_i \in [0,\pi)$, every polynomial $P \in \Pi_n(\mathbb{R}^2)$ can be represented as a sum of $n+1$ ridge polynomials $Q_i(\theta_i, x, y) = Q_i(x\cos\theta_i + y\sin\theta_i)$, $i = 0, \ldots, n$.*

This is a known result (see [6]). For the sake of completeness, we supply here a new simple proof.

Proof. The functions $(t-t_0)^m, \ldots, (t-t_m)^m, t_i \neq t_j, i \neq j$, are linearly independent. Since their number is $m+1$ they form a basis of $\Pi_m(\mathbb{R})$. Then the following representation holds

$$t^k = \sum_{j=0}^{m} \alpha_j^k (t-t_j)^m,$$

with appropriate coefficients $\{\alpha_j^k\}$.

$$x^k y^{m-k} = \sum_{j=0}^{m} \alpha_j^k (x - yt_j)^m = \sum_{j=0}^{m} \alpha_j^k (\overline{a}_j \cdot \overline{x})^m,$$

where $\overline{x} = (x, y)$ and $\overline{a}_j = (1, -t_j)$ and $\overline{a}_j \cdot \overline{x}$ is the dot product of the vectors $\overline{a}_j$ and $\overline{x}$, $j = 0, \ldots m$. Clearly $(\overline{a}_j \cdot \overline{x})^m$ is a ridge polynomial of degree m. Then the polynomial

$$P(x, y) = \sum_{m=0}^{n} \sum_{k=0}^{m} a_{mk} x^k y^{m-k}$$

can be written as

$$P(x, y) = \sum_{m=0}^{n} \sum_{k=0}^{m} a_{mk} \sum_{j=0}^{m} \alpha_j^k (\overline{a}_j \overline{x})^m.$$

Grouping in the proper way shows that $P(x, y)$ can be represented in the form

$$P(\overline{x}) = f_0(\overline{a}_0 \overline{x}) + f_1(\overline{a}_1 \overline{x}) + \cdots + f_n(\overline{a}_n \overline{x}),$$

that is, as a linear combination of the ridge polynomials f_i of degree not greater than n.

Theorema 2.2. *Every algebraic polynomial $f \in \Pi_n(\mathbb{R}^2)$ is determined by $n + 1$ projections*

$$P_f(t, \theta_0), \ldots, P_f(t, \theta_n), \ \theta_i \neq \theta_j, i \neq j.$$

P r o o f. Since

$$f(x, y) = \sum_{i+j \leq n} \left\{ a_{ij} x^i y^j \right\},$$

we have

$$P_f(t, \theta) = \sum_{i+j \leq n} a_{ij} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} (t \cos\theta - s \sin\theta)^i (t \sin\theta + s \cos\theta)^j \, ds.$$

From the condition $i + j \leq n$ it follows that the integrand is a trigonometric polynomial of θ of order n and after integration on s it remains a polynomial of the same kind. Since $P_f(t, \theta + \pi) = P_f(-t, \theta)$, the projection $P_f(t, \theta)$ is known for $2n + 2$ different values of θ in the interval $[0, 2\pi)$. Let us mention that every trigonometric polynomial is determined by its values in $2n + 1$ points in $[0, 2\pi)$. It follows from this property that $P_f(t, \theta)$ can be recovered for every θ. Finally, using Radon theorem we conclude that f is determined uniquely by $P_f(t, \theta_0), \ldots, P_f(t, \theta_n.)$

Since a function, identically equal to zero, has zero projections $P(t, \theta_i)$, using the above theorem we get:

Corollary 2.1. *Let* $f \in \Pi_n(\mathbb{R}^2)$. *If* $P_f(t, \theta) = 0$ *for* $\theta_0, \ldots, \theta_n \in [0, \pi)$, $\theta_i \neq \theta_j, i \neq j$, *then* $f = 0$.

Let us denote by $U_r(t)$ the Tchebicheff polynomial of the second kind of degree r, i. e.

$$U_r(t) = \frac{1}{\sqrt{\pi}} \frac{\sin(n+1)\arccos t}{\sqrt{1-t^2}}.$$

Consider the ridge polynomials

$$U_{rl}(x, y) := U_r\left(x \cos\frac{l\pi}{r+1} + y \sin\frac{l\pi}{r+1}\right).$$

The equality $\int\int_B U_{rl}^2(x, y)\, dx\, dy = 1$ explains why in the formula for $U_r(t)$ we take a multiplayer $\frac{1}{\sqrt{\pi}}$.

By Marr's formula [7],

$$\int_{I(t,\theta)} U_{rl}(x, y)\, ds = 2\frac{\sqrt{1-t^2}}{r+1} U_r(t) \frac{\sin(r+1)(\theta - \frac{l\pi}{r+1})}{\sin(\theta - \frac{l\pi}{r+1})}.$$

A different proof can be seen also in [3], Lemma 5. This formula was used in [3] to derive the following

Theorema 2.3. *The ridge polynomials* $U_{rl}(x, y)$, $l=0,\ldots,n$, $r=0,\ldots,n$ *form an orthonormal basis of* $\Pi_n(\mathbb{R}^2)$.

This theorem was established earlier by Logan and Shepp [6].

3. First method for reconstruction

3.1 Recovery of polynomials, based on Radon projections

First we consider the following

Reconstruction problem 3.1. *Given are* $\theta_0, \ldots, \theta_n, \theta_i \neq \theta_j, i \neq j, \theta_i \in (0, \pi)$ *and functions* $f_i(t)$, $i = 0, \ldots, n$, *such that they are projections of some polynomial* $f \in \Pi_n(\mathbb{R}^2)$ *along the lines*

$$I(t, \theta_i) = \{(x, y) : x\cos\theta_i + y\sin\theta_i = t\}, \; t \in [-1, 1].$$

The problem is to reconstruct f on the unit disk.

We are going to search f in the form

$$f = \sum_{r=0}^{n} \sum_{l=0}^{r} a_{rl} U_{rl}.$$

This is possible since by Theorem 1.3 $\{U_{rl}\}_{l=0}^{r}{}_{r=0}^{n}$ form a basis of $\Pi_n(\mathbb{R}^2)$. Using Marr's formula for the projection of U_{rl}, we get

$$f_i(t) = \int_{I(t,\theta_i)} f(x,y)\, ds = \sum_{r=0}^{n} \sum_{l=0}^{r} a_{rl} \int_{I(t,\theta_i)} U_{rl}(x,y)\, ds$$

$$= \sum_{r=0}^{n} \sum_{l=0}^{r} a_{rl} 2 \frac{\sqrt{1-t^2}}{r+1} U_r(t) \frac{\sin(r+1)(\theta_i - \frac{l\pi}{r+1})}{\sin(\theta_i - \frac{l\pi}{r+1})}, \quad i = 0, \ldots, n.$$

Let us introduce the functions

$$g_i(t) := \sum_{r=0}^{n} \sum_{l=0}^{r} a_{rl} \frac{2}{r+1} U_r(t) \frac{\sin(r+1)(\theta_i - \frac{l\pi}{r+1})}{\sin(\theta_i - \frac{l\pi}{r+1})}.$$

Clearly $f_i(t) = \sqrt{1-t^2} g_i(t)$. Since $g_i(t) \in \Pi_n(\mathbb{R})$, it can be represented it in the form

$$g_i(t) = \sum_{r=0}^{n} A_r^i U_r(t),$$

where the quantities

$$A_r^i = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-t^2} g_i(t) U_r(t)\, dt = \frac{2}{\pi} \int_{-1}^{1} f_i(t) U_r(t)\, dt$$

are known. These two representations of $g_i(t)$ give

$$A_r^i = \frac{2}{r+1} \sum_{l=0}^{r} a_{rl} \frac{\sin(r+1)(\theta_i - \frac{l\pi}{r+1})}{\sin(\theta_i - \frac{l\pi}{r+1})}.$$

Using the notations $B_r^i = \frac{r+1}{2} A_r^i$, we rewrite the last integral as

$$B_r^i = \sum_{l=0}^{r} a_{rl} \frac{\sin(r+1)(\theta_i - \frac{l\pi}{r+1})}{\sin(\theta_i - \frac{l\pi}{r+1})}.$$

Taking it for $i = 0, \ldots, r$ we arrive at a linear system of equations in unknown $a_{r0}, \ldots, a_{rr}$.

We shall show next that the system for any given data $\{B_r^i\}$. To do this, is suffices to prove that its determinant Δ is non-zero. Let us denote by $\Delta_n^{(n-k)}(\theta)$ the following matrix

$$\Delta_n^{(n-k)}(\theta) = \begin{bmatrix} \frac{\sin(n+1)(\theta-\frac{k\pi}{n+1})}{\sin(\theta-\frac{k\pi}{n+1})} & \frac{\sin(n+1)(\theta-\frac{(k+1)\pi}{n+1})}{\sin(\theta-\frac{(k+1)\pi}{n+1})} & \cdots & \frac{\sin(n+1)(\theta-\frac{n\pi}{n+1})}{\sin(\theta-\frac{n\pi}{n+1})} \\ \frac{\sin(n+1)(\theta_{k+1}-\frac{k\pi}{n+1})}{\sin(\theta_{k+1}-\frac{k\pi}{n+1})} & \frac{\sin(n+1)(\theta_{k+1}-\frac{(k+1)\pi}{n+1})}{\sin(\theta_{k+1}-\frac{(k+1)\pi}{n+1})} & \cdots & \frac{\sin(n+1)(\theta_{k+1}-\frac{n\pi}{n+1})}{\sin(\theta_{k+1}-\frac{n\pi}{n+1})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sin(n+1)(\theta_n-\frac{k\pi}{n+1})}{\sin(\theta_n-\frac{k\pi}{n+1})} & \frac{\sin(n+1)(\theta_n-\frac{(k+1)\pi}{n+1})}{\sin(\theta_n-\frac{(k+1)\pi}{n+1})} & \cdots & \frac{\sin(n+1)(\theta_n-\frac{n\pi}{n+1})}{\sin(\theta_n-\frac{n\pi}{n+1})} \end{bmatrix}$$

for $k = 0, \ldots, n-1$.

Note that the determinant of the system above for $\{a_{rl}\}$ is of the kind

$$\Delta = \begin{vmatrix} 1 & & & 0 \\ & \Delta_1(\theta_0) & & \\ & & \Delta_2(\theta_0) & \\ & & \ddots & \vdots \\ 0 & & \cdots & \Delta_n(\theta_0) \end{vmatrix},$$

where $\Delta_n(\theta) = \Delta_n^{(n)}(\theta)$.

Let us mention that $\Delta = 1.det\Delta_1(\theta_0)\ldots det\Delta_n(\theta_0)$. Evidently $\Delta \neq 0$. Since $\frac{\sin(n+1)\alpha}{\sin\alpha}$ is a trigonometric polynomial of order n, then $det\Delta_n(\theta)$ is also a trigonometric polynomial of order n.

Observe that $det\Delta_n(\theta)$ vanishes at $\theta_1, \ldots, \theta_n, \theta_1 + \pi, \ldots, \theta_n + \pi$. We shall show that $det\Delta_n(\theta)$ is not identically equal to zero. Indeed, suppose that $det\Delta_n(0) = 0$. But

$$det\Delta_n(0) = det\Delta_n^{(n)}(0) = (n+1)det\Delta_n^{(n-1)}(\theta_1).$$

Our next task is to prove that $det\Delta_n^{(n-1)}(\theta_1) \neq 0$. To do this, note that $det\Delta_n^{(n-1)}(\theta)$ vanishes at $\theta_2, \ldots, \theta_n, \theta_2+\pi, \ldots, \theta_n+\pi, 0, \pi$. Hence it has $2n$ zeros. Since θ_1 differs from all these zeros it is enough to show that $det\Delta_n^{(n-1)}(\theta)$ is not identically zero. So we need some θ such that $det\Delta_n^{(n-1)}(\theta) \neq 0$. Let us take $\theta = \frac{\pi}{n+1}$. In this way we get

$$\Delta_n^{(n-1)}\left(\frac{\pi}{n+1}\right) = \begin{bmatrix} (n+1) & 0 \ldots 0 \\ \star & \Delta_n^{(n-2)}\left(\frac{\pi}{n+1}\right) \end{bmatrix}.$$

$Det\Delta_n^{(n-2)}(\theta)$ vanishes at $\theta_3, \dots, \theta_n, \theta_3+\pi, \dots, \theta_n+\pi, 0, \pi, \frac{\pi}{n+1}, \pi - \frac{\pi}{n+1}$. Hence it has $2n$ zeros. So we need some θ such that $det\Delta_n^{(n-2)}(\theta) \neq 0$. We take $\theta = \frac{2\pi}{n+1}$ and continue this process until we get to the factor

$$\begin{vmatrix} \frac{\sin(n+1)(\theta_{n-1}-\frac{(n-1)\pi}{n+1})}{\sin(\theta_{n-1}-\frac{(n-1)\pi}{n+1})} & \frac{\sin(n+1)(\theta_{n-1}-\frac{(n-1)\pi}{n+1})}{\sin(\theta_{n-1}-\frac{(n-1)\pi}{n+1})} \\ \frac{\sin(n+1)(\theta_{n}-\frac{(n-1)\pi}{n+1})}{\sin(\theta_{n}-\frac{(n-1)\pi}{n+1})} & \frac{\sin(n+1)(\theta_{n}-\frac{n)\pi}{n+1})}{\sin(\theta_{n}-\frac{n\pi}{n+1})} \end{vmatrix}$$

$$= \sin(n+1)\theta_{n-1}\sin(n+1)\theta_n \sin\frac{\pi}{n+1}\sin(\theta_{n-1}-\theta_n),$$

which is obviously different from 0. Hence $\Delta \neq 0$.

The reconstruction of a bivariate polynomial $f(x,y)$ on the basis of $n+1$ Radon projections can be done also in the following way in which, instead of solving a linear system one is led up to calculating univariate integrals. Let us describe it in brief. Since $P_f(t, \theta_i+\pi) = P_f(-t, \theta_i)$, we can suppose that $P_f(t,\theta)$ is known at $2n+2$ values of θ. The projection $P_f(t,\theta)$ is a trigonometric polynomial of θ of order n, hence we can assume that it is known. We have

$$a_{rl} = \int\int_D f(x,y)U_{rl}(x,y)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{I(t,\frac{l\pi}{r+1})} fU_{rl}\,ds\,dt = \int_{-1}^{1} U_r(t)P_f(t,\frac{l\pi}{r+1})\,dt.$$

Computing the last integrals one gets the coefficients. Then

$$f = \sum_{r=0}^{n}\sum_{l=0}^{r} a_{rl}U_{rl}.$$

3.2 An algorithm for reconstruction

On the basis of the results from the previous section we suggest an algorithm for reconstruction of a function f from its Radon projections.

Suppose that $\theta_0, \dots, \theta_n$, are given angles, $\theta_i \neq \theta_j, i \neq j, \theta_i \in (0,\pi)$, and the functions $f_i(t)$, $i = 0, \dots, n$, are known to be the projections of a certain function f along the chords $I(t,\theta_i) = \{(x,y) : x\cos\theta_i + y\sin\theta_i = t\}$, $t \in [-1,1]$. Using this information we shall construct the polynomial

$$S_n(f;x,y) = \sum_{r=0}^{n}\sum_{l=0}^{r} a_{rl}U_{rl}(x,y),$$

where $a_{rl} =< f, U_{rl} >:= \int\int_B fU_{rl}$. We shall show that $S_n(f)$ approximates f.

Assume that f admits a representation of the form

$$f = \sum_{r=0}^{\infty}\sum_{l=0}^{r} a_{rl}U_{rl}.$$

Then

$$f_i(t) = \sum_{r=0}^{\infty}\sum_{l=0}^{r} a_{rl}\int_{I(t,\theta_i)} U_{rl}(x,y)\,ds$$

$$= \sum_{r=0}^{\infty}\sum_{l=0}^{r} a_{rl}2\frac{\sqrt{1-t^2}}{r+1}U_r(t)\frac{\sin(r+1)(\theta_i-\frac{l\pi}{r+1})}{\sin(\theta_i-\frac{l\pi}{r+1})}$$

$$= \sqrt{1-t^2}\sum_{r=0}^{\infty}U_r(t)\sum_{l=0}^{r} a_{rl}\frac{2}{r+1}\frac{\sin(r+1)(\theta_i-\frac{l\pi}{r+1})}{\sin(\theta_i-\frac{l\pi}{r+1})},\ i=0,\dots,n.$$

On the other hand, we can represent $f_i(t)$ in the form

$$f_i(t) = \sqrt{1-t^2}\sum_{r=0}^{\infty}A_r^iU_r(t),$$

where $A_r^i = \frac{2}{\pi}\int_{-1}^{1} f_i(t)U_r(t)\,dt$. Comparing the coefficients in the both representations of f_i, we get as in the polynomial case that

$$A_r^i = \frac{2}{r+1}\sum_{l=0}^{r} a_{rl}\frac{\sin(r+1)(\theta_i-\frac{l\pi}{r+1})}{\sin(\theta_i-\frac{l\pi}{r+1})}.$$

Given $\{f_i\}_0^n$, we can calculate $\{A_r^i\}$ and find the functions

$$f_{i,n}(t) = \sqrt{1-t^2}\sum_{r=0}^{n}A_r^iU_r(t),\ i=0,\dots,n.$$

Using the above algorithm for polynomials we find the unique polynomial F_n from $\Pi_n^2(\mathbb{R}^2)$, such that

$$(*)\qquad f_{i,n}(t) = \int_{I(t,\theta_i)} F_n(x,y)\,ds,\ i=0,\dots,n.$$

Consider the polynomial $S_n(f;x,y)$. We have

$$\int_{I(t,\theta_i)} S_n(f;x,y)\,ds = \sum_{r=0}^{n}\sum_{l=0}^{r} a_{rl}\int_{I(t,\theta_i)} U_{rl}$$

$$= \sum_{r=0}^{n}\sum_{l=0}^{r} a_{rl} 2\frac{\sqrt{1-t^2}}{r+1} U_r(t) \frac{\sin{(r+1)(\theta_i - \frac{l\pi}{r+1})}}{\sin{(\theta_i - \frac{l\pi}{r+1})}}$$

$$= \sqrt{1-t^2} \sum_{r=0}^{n} U_r(t) \sum_{l=0}^{r} a_{rl} \frac{2}{r+1} \frac{\sin{(r+1)(\theta_i - \frac{l\pi}{r+1})}}{\sin{(\theta_i - \frac{l\pi}{r+1})}}$$

$$= \sqrt{1-t^2} \sum_{r=0}^{n} U_r(t) A_r^i = f_{i,n}(t).$$

Thus the Radon projections of $S_n(f;x,y)$ coincide with those of F_n. By the results of previous section this yields that $S_n(f;x,y) \equiv F_n(x,y)$.

Now, let us turn to the question about the rate of approximation of f by $S_n(f)$. We shall use a result by Oskolkov, namely Theorem 3 from [8]:

Theorem 3.1. *Every function $f \in L_2(B)$ can be uniquely represented by the corresponding Tchebicheff–Fourier series*

$$f(x,y) = \sum_{r=0}^{\infty}\sum_{l=0}^{r} a_{rl} U_{rl}(x,y),$$

where

$$a_{rl} = \int\int_D f(x,y) U_{rl}(x,y)\, dx\, dy,$$

and this series converges in $L_2(B)$.

Since $S_n(f)$ is the n–th partial sum of the Tchebicheff–Fourier series of f, Theorem 3.1 implies the polynomial $S_n(f)$ tends to the function f in L_2–norm when $n \to \infty$.

We are interested in the rate of the uniform convergence of $S_n(f)$ to f when $n \to \infty$. The operator S_n acts from $C(B)$ to $C(B)$. Recall that the Lebesgue constant of S_n is defined by

$$\|S_n\|_{C(B)\to C(B)} = \max_{x\in B} \int_B |\sum_{r=0}^{n}\sum_{l=0}^{r} U_{rl}(x) U_{rl}(t)|\, dt,$$

and for $f \in C(B)$, $\omega(f)$ denotes the modulus of continuity of f:

$$\omega(f,h) = \sup_{x,\ y\in B\ |x-y|\le h} \|f(x) - f(y)\|_{C(B)}.$$

We shall use these notations in our estimate. We follow the ideas from [10] where the author considers a orthogonal polynomial system $\{P_{rl}\}_{r,\ l}$ in $L_{2,w}$, with respect to the weight $w(x) = \pi^{-1}(1-|x|)^{1/2}$ for $\overline{x} \in B$.

In the notations of [10],

$$s_n(f) = \sum_{r=0}^{n} < f, P_{rl} > P_{rl}.$$

The theorem for the rate of convergence of $s_n(f)$, proved by Skopina in [Sk] says:

"Let $f \in C(B)$, $\overline{x} \in B$ then

$$|f(\overline{x}) - s_n(f, \overline{x})| \leq A\sqrt{n}\omega\left(f, \frac{1}{n}\right),$$

where A is an absolute constant. In particular if $f \in Lip\ \alpha$, $\alpha > \frac{1}{2}$, then the Fourier series $\sum_{r=0}^{\infty} < f, P_{rl} > P_{rl}$ converges uniformly to f on B."

The proof is based on some facts about the relation between $s_n(f)$ and the partial sums of Laplace series of $F \in C(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$. Here the associated function F to $f \in C(B)$ is defined by $F(x, y, z) = f(x, y)$ for all $(x, y) \in S^2$. In our case $\{U_{rl}\}$ is the orthogonal system with respect to the weight $\equiv 1$. The corresponding partial sums of the Laplace series in our case look like:

$$\tilde{\sigma}_n(F, x) = \frac{1}{2\pi} \sum_{k=0}^{n} (k+1) \int_{S^2} F(t) U_k(t \cdot x) \sqrt{1 - t^2}\, ds(t).$$

Analogously to the proof of theorem 3 in [10] we can show that for $f \in C(B)$,

$$\tilde{\sigma}_n(F, \overline{x}) = S_n(f; \overline{x})$$

for all $\overline{x} \in S^2$.

In our case the corresponding estimate for the Lesbegue constant is

$$\|S_n\|_{C(B)\to C(B)} \leq \int_{-1}^{1} \left| \sum_{k=0}^{n} U_k(1) U_k(\tau) \right| \sqrt{1 - \tau^2}\, d\tau.$$

For the following theorem we use ideas from the proof of Theorem 6 in [10], namely Jackson's theorem and some results of [1] on the behaviour of the last integral when n tends to infinity.

In our terms,

$$U_n(x) = \frac{1}{\sqrt{2}} \frac{\Gamma(n+2)}{\Gamma(n+3/2)} P_n^{\frac{1}{2},\frac{1}{2}}(x)$$

and the integral

$$L_n^{\frac{1}{2},\frac{1}{2}}(x) = \int_{-1}^{1} \left| \sum_{k=0}^{n} h_k P_k(t) P_k(x) \right| p(t)\, dt$$

takes the form

$$\int_{-1}^{1} \left| 2h_k \frac{\Gamma^2(k+3/2)}{\Gamma^2(k+2)} U_k(t) U_k(x) \right| \sqrt{1-t^2}\, dt.$$

We can apply the main result of [1] for

$$h_k = \frac{\Gamma^2(k+2)}{2\Gamma^2(k+3/2)} \sim 2^{-\frac{1}{2}-\frac{1}{2}} + O(1)$$

since

$$\lim_{k\to\infty} \frac{\Gamma(k+2)}{\Gamma(k+3/2)\sqrt{k}} = 1.$$

We obtain

$$L_n^{\frac{1}{2},\frac{1}{2}}(x) = \int_{-1}^{1} \left| \sum_{k=0}^{n} h_k U_k(t) U_k(x) \right| \sqrt{1-t^2}\, dt \le C\sqrt{n}.$$

As in [10], this leads to the following conclusion.

Theorema 3.2. *Let $f \in C(B)$, $\overline{x} \in B$. Then*

$$|f(\overline{x}) - S_n(f, \overline{x})| \le A\sqrt{n}\omega\left(f, \frac{1}{n}\right),$$

where A is an absolute constant. In particular if $f \in Lip\ \alpha$, $\alpha > \frac{1}{2}$, then the Tchebicheff–Fourier series $\sum_{r=0}^{\infty} < f, U_{rl} > U_{rl}$ converges uniformly to f on B.

Next we illustrate the algorithm on two examples.

The first one is an example on which usually the new methods for recovery of surfaces are tested. The next example is the function $\sin x \cos 2y$. Since this function is smooth, the convergence of the approximating polynomial is faster than in the first example.

Let us mention a property of the interpolating polynomial $S_n(f; x, y)$.

Theorem 3.3. *Let $f(x,y) = f(-x,y)$. Then $S_n(f;x,y) = S_n(f;-x,y)$.*

P r o o f. Let us note that

$$U_{rl}(-x,y) = U_r\left(-x\cos\frac{l\pi}{r+1} + y\sin\frac{l\pi}{r+1}\right)$$

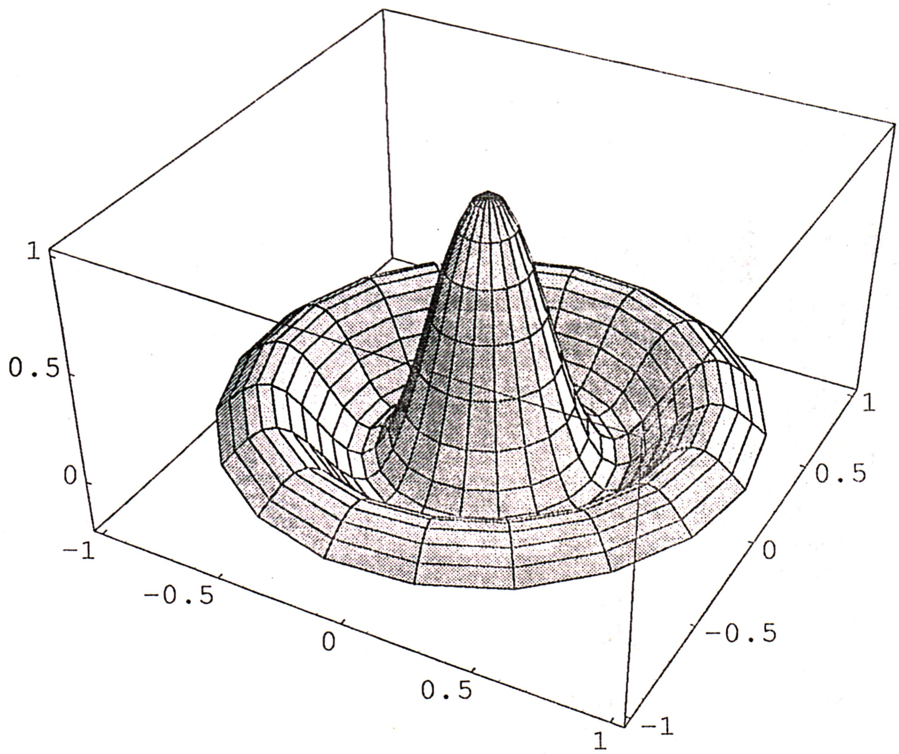

Figure 1: The graph of the function Mexican hat $\frac{\sin(3\pi\sqrt{x^2+y^2+10^{-18}})}{3\pi\sqrt{x^2+y^2+10^{-18}}}$.

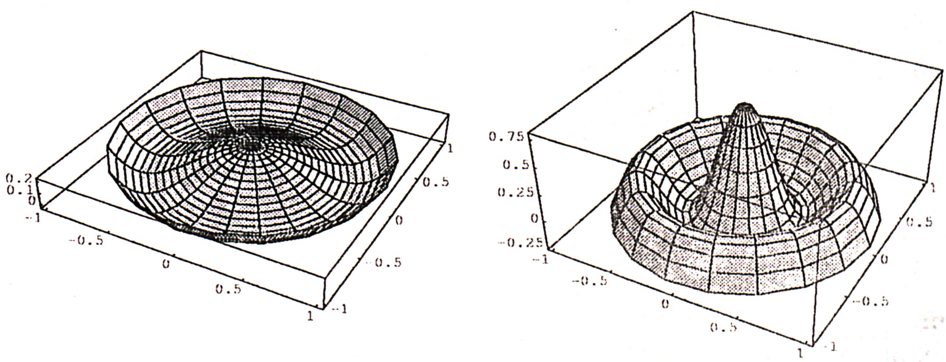

Figure 2: The graph of the approximating polynomial $\in \Pi_4(\mathbb{R}^2)$ and the graph of the error $\in \Pi_4(\mathbb{R}^2)$.

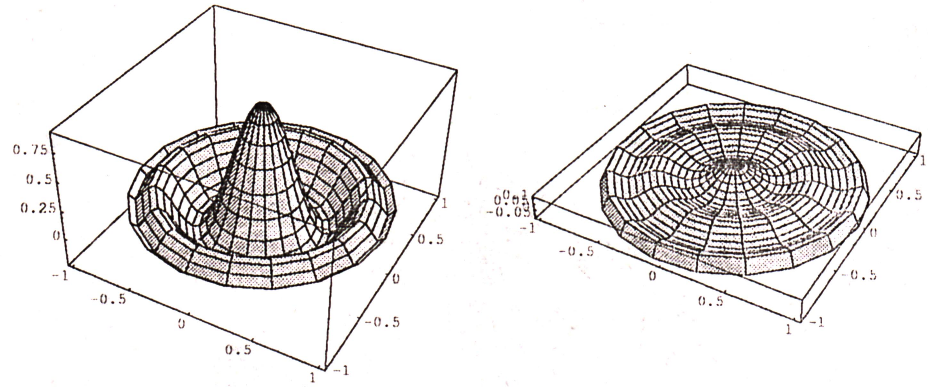

Figure 3: The graph of the approximating polynomial $\in \Pi_8(\mathbb{R}^2)$ and the graph of the error $\in \Pi_8(\mathbb{R}^2)$.

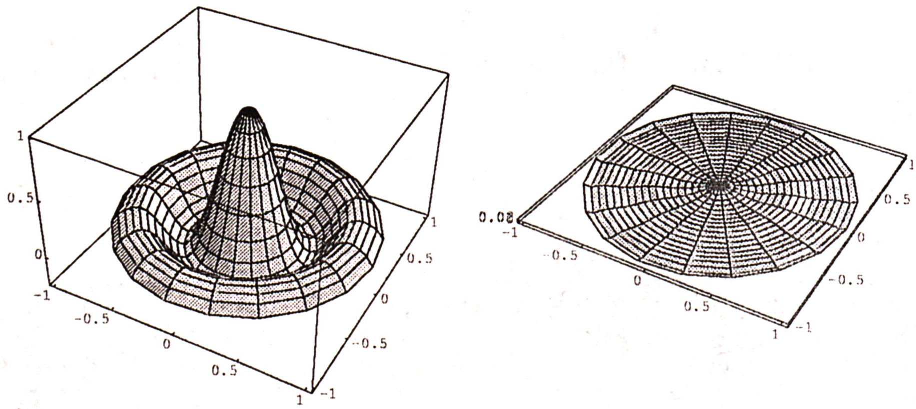

Figure 4: The graph of the approximating polynomial in $\in \Pi_{10}(\mathbb{R}^2)$ and the graph of the error $\in \Pi_{10}(\mathbb{R}^2)$.

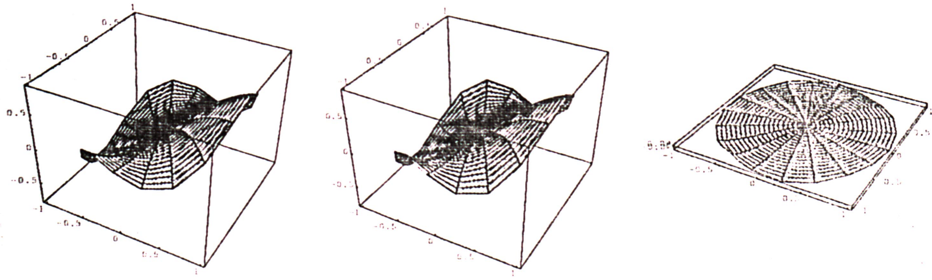

Figure 5: The graphs of the function $\sin x \cos 2y$, its approximating polynomial $\in \Pi_3(\mathbb{R}^2)$ and the graph of the error.

$$= U_r\left(-x(-1)\cos(\pi - \frac{l\pi}{r+1}) + y\sin(\pi - \frac{l\pi}{r+1})\right)$$

$$= U_r\left(x\cos\frac{r+1-l}{r+1}\pi + y\sin\frac{r+1-l}{r+1}\pi\right) = U_{r,r+1-l}(x,y),$$

for l=1,...,r. Then

$$a_{rl} = \int\int_D f(x,y)U_{rl}(x,y)\,dx\,dy$$

$$= \int\int_D f(-x,y)U_{r,r+1-l}(-x,y)\,dx\,dy = a_{r,r+1-l},$$

for l=1,...,r. Moreover,

$$a_{r0} = \int\int_D f(x,y)U_{r0}(x,y)\,dx\,dy = \int\int_D f(-x,y)U_{r0}(-x,y)\,dx\,dy$$

$$= \int\int_D f(x,y)(-1)^r U_{r0}(x,y)\,dx\,dy = (-1)^r a_{r0},$$

because

$$U_{r0}(-x,y) = U_r(-x) = (-1)^r U_r(x) = (-1)^r U_{r0}(x,y).$$

Therefore,

$$\begin{aligned} S_n(f;-x,y) &= \sum_{r=0}^{n}\sum_{l=0}^{r} a_{rl}U_{rl}(-x,y) = \sum_{r=0}^{n}\left(a_{r0}U_{r0}(-x,y) + \sum_{l=1}^{r} a_{rl}U_{rl}(-x,y)\right) \\ &= \sum_{r=0}^{n}\left((-1)^r a_{r0}(-1)^r U_{r0}(x,y) + \sum_{l=1}^{r} a_{r,r+1-l}U_{r,r+1-l}(x,y)\right) \\ &= \sum_{r=0}^{n}\left(a_{r0}U_{r0}(x,y) + \sum_{k=1}^{r} a_{rk}U_{rk}(x,y)\right) = S_n(f;x,y). \end{aligned}$$

4. Second method for reconstruction

Consider the following: Suppose that $M_i(\cos\alpha_i, \sin\alpha_i)$, $i = 0, \dots, n$, are fixed distinct points on the unit circle ∂B. Let

$$J(M_i, P) := \int_{\overline{M_i P}} f\, ds, \; P \in \partial B, \quad i = 0, \dots, n$$

be the Radon projections of a certain polynomial $f \in \Pi\left(\mathbb{R}^2\right)$, (for all $P \in \partial B$). Find the polynomial f. We prove the following

Theorem 4.1. *The polynomial $f \in \Pi_n\left(\mathbb{R}^2\right)$ can be uniquely reconstructed from the information*

$$\int_{\overline{M_i P}} f\, ds, \; i = 0, \dots, n, \; P \in \partial B.$$

Proof. Take a point $M(\cos\alpha, \sin\alpha)$, on the unit circle ∂B, different from any of the points M_i. It follows from the assumptions of the theorem that the values

$$(*) \quad \begin{aligned} &J(M_i, M) = \gamma_i\,, && i = 0, \ \dots, \ n \\ &J(M_i, M_j) = \gamma_k\,, && k = n+1, \dots, \tfrac{(n+1)(n+2)}{2} \end{aligned}$$

are known. The conditions of Hakopian's theorem [2] are fulfilled, hence there exists a polynomial $f_M \in \Pi\left(\mathbb{R}^2\right)$ such that

$$\int_{\overline{M_i M}} f_M\, ds = \gamma_i, \; i = 0, \dots, n$$

and

$$\int_{\overline{M_i M_j}} f_M\, ds = \gamma_k, \; k = n+1, \dots, \frac{(n+1)(n+2)}{2}.$$

Since $\int_{\overline{M_i P}} f\, ds$ are projections of the polynomial f, then f satisfies the conditions (*). Using the uniqueness of Hakopian's theorem we conclude that f coincides with f_M. This proves the theorem.

Our next purpose is to find out an explicit form for the solution of the interpolation problem. We are going to apply an idea which was suggested to us from H. Hakopian [4] for a simple proof of his interpolation theorem in the bivariate case.

It is more convenient to denote the point M by M_{n+1}. In this way we get $(n+2)$ points $M_i(\cos\alpha_i, \sin\alpha_i)$, $i = 0, \dots, n+1$, on the unit circle. Connecting any two of these points by line segments we get $\frac{(n+2)(n+1)}{2}$ chords:

$$I_j = I(t_j, \theta_j) = \{(x, y) : \; x\cos\theta_j + y\sin\theta_j = t_j\}, \; t_j \in [-1, 1].$$

Suppose that the points M_i are such that $\theta_i \neq \theta_j, i \neq j,\ \theta_i - \theta_j \neq \pi\ , i \neq j,\ \theta_j \in [0, 2\pi),\ j = 0, \dots, (n+1)$. Let $I(t_j, \theta_j)$ be the chord determined by the points M_p and M_q. The quantity

$$\int_{I(t_j,\theta_j)} f = \int_{\overline{M_p M_q}} f$$

is known, we denote it by c_j. We search f in the form

$$\sum_{k=1}^{\frac{(n+2)(n+1)}{2}} c_k P_k(x\cos\theta_k + y\sin\theta_k),$$

where $P_k \in \Pi_n(\mathbb{R})$ is such that the integral of the ridge polynomial $P_k(x\cos\theta_k + y\sin\theta_k)$ over I_j is equal to δ_{jk}.

Let us fix the k−th chord I_k and introduce a coordinate system with origin at the zero and parallel to I_k. (Let us mention that $\theta_k = 0$ because of the choice of the coordinate system.) After calculating t_j and θ_j we get $t_j = \cos\frac{\alpha_p - \alpha_q}{2}$ and $\theta_j = \frac{\alpha_p + \alpha_q}{2}$. Then the chord $M_p M_q$ has the following parametrisation:

$$x = \cos\frac{\alpha_p - \alpha_q}{2}\cos\frac{\alpha_p + \alpha_q}{2} - s\sin\frac{\alpha_p + \alpha_q}{2}$$

$$y = \cos\frac{\alpha_p - \alpha_q}{2}\sin\frac{\alpha_p + \alpha_q}{2} + s\cos\frac{\alpha_p + \alpha_q}{2},$$

where

$$-\left|\sin\frac{\alpha_p - \alpha_q}{2}\right| \le s \le \left|\sin\frac{\alpha_p - \alpha_q}{2}\right|.$$

Let $j \neq k$. Then

$$0 = \int_{I(t_j,\theta_j)} P_k(x\cos\theta_k + y\sin\theta_k)\, ds$$

$$= \int_{-\infty}^{+\infty} P_k((t_j\cos\theta_j - s\sin\theta_j)\cos\theta_k + (t_j\sin\theta_j + s\cos\theta_j)\sin\theta_k)\, ds$$

$$= \int_{-\sqrt{1-t_j^2}}^{\sqrt{1-t_j^2}} P_k((t_j\cos(\theta_j - \theta_k) - s\sin(\theta_j - \theta_k))\, ds.$$

After the change of variable $v = t_j\cos(\theta_j - \theta_k) - s\sin(\theta_j - \theta_k),\ dv = -\sin(\theta_j - \theta_k)\, ds$, let us set

$$x_j^1 = t_j\cos(\theta_j - \theta_k) + \sqrt{1-t_j^2}\sin(\theta_j - \theta_k),$$

$$x_j^2 = t_j \cos(\theta_j - \theta_k) - \sqrt{1 - t_j^2} \sin(\theta_j - \theta_k).$$

Since $x = t\cos\theta - s\sin\theta$, then one of the end points of I_j is projected into x_j^1, the other- into x_j^2. After the change of variable we get $0 = \int_{x_j^1}^{x_j^2} P_k(v)\, dv$. The number of the points M_i is $(n+2)$ but the two end points of I_k have the same projection so we get $(n+1)$ points which we order by size and call $x_0, \dots, x_n$. The condition $\theta_i \neq \theta_j, i \neq j,\ \theta_i - \theta_j \neq \pi$, $i \neq j$, provides that the points $x_0, \dots, x_n$ are different.

It follows from the interpolation conditions on P_k that $\int_{x_i}^{x_{i+1}} P_k(v)\, dv = 0,\ i = 0, \dots, n-1$.

The condition $\int_{I(\theta_k, t_k)} P_k(x\cos\theta_k + y\sin\theta_k)\, ds = 1$ gives

$$1 = \int_{-\sqrt{1-t_k^2}}^{\sqrt{1-t_k^2}} P_k(t_k \cos^2\theta_k + t_k \sin^2\theta_k)\, ds = P_k(t_k) 2\sqrt{1 - t_k^2}.$$

Let us introduce the polynomial $Q_k(x) = \int_{x_0}^{x} P_k(t)\, dt$. Then the conditions on P_k pass into

$$Q_k(x_0) = 0,\ \dots,\ Q_k(x_n) = 0,\ Q_k'(t_k) = P_k(t_k) = \frac{1}{2\sqrt{1 - t_k^2}}.$$

After finding $Q_k(x)$ out, we get $P_k(x) = Q'(x)$.

The desired ridge polynomial is $P_k(x\cos\theta_k + y\sin\theta_k)$.

As in the previous section, the solution of the polynomial problem can be used to construct an algorithm for approximation of functions, based on the projections $\left\{ \int_{\overline{M_i M_j}} f\, ds \right\}$.

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