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Integral Primary Decomposition of Differential Modules

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If M is a differential module and the factor-module $M / \int_M^\omega 0$ has no differential submodules other than 0 and M, the zero submodule is called ω – primary. In this paper we prove the existence and uniqueness of an ω – primary decomposition of a submodule of a differentially noetherian module.

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Throughout the paper R will denote an associative ring with identity, Δ is a set of derivations on A. A derivation $\delta \in \Delta$ is said to be nil if for each $x \in R$ there is a natural number $n = n(x)$ such that $\delta^n(x) = 0$. Let all derivaions from Δ are nil. In [1] L. O. Chung proved the results for a prime ring with one nil derivation. In [2] the present author have the results for a Ritt algebra with one nil derivation. We assume that the constant subring $C_R = \bigcap_{\delta\in\Delta} \{x \in R | \delta(x) = 0\}$ of R is included in the centre of R. The free commutative semigroup generated by Δ will be denoted by Θ, the set $\mathfrak{D}_R$ of all finite linear combinations of elements from Θ over R be the ring of differential operators on R and $\mathfrak{D}_{C_R}$ be the subring of differential operators with constant coefficients.

The left $\mathfrak{D}_R$ – modules are called differential modules. For differential modules A and B, $B \subseteq A$, and $\omega \in \mathfrak{D}_{C_R}$ in [3] we define

$$\int_A^\omega B = \{x | x \in A, \exists n \in \mathbb{N} \cup \{0\}, \omega^n(x) \in B\}.$$

Let $\mathfrak{L}$ be the lattice of differential submodules of a fixed differential module M and

$$\mathcal{A} = \left\{(A, B) | A, B \in \mathfrak{L}, B \subseteq A\right\}.$$

Having the results of [3] and [4] we immediately receive

Proposition 1. *The mapping* $I^\omega : \mathcal{A} \longrightarrow \mathcal{L}$, *defined by* $I^\omega(A,B) = \int_A^\omega B$, *where* $(A,B) \in \mathcal{A}$, *is a strongly hereditary radical closure in the lattice* $\mathcal{A}$.

An immediate consequence from the Proposition 1 is the equality

$$\left(\int_A^\omega B\right) \cap \left(\int_{A_1}^\omega B_1\right) = \int_{A\cap A_1}^\omega (B \cap B_1) \text{ for arbitrary } (A,B),(A_1,B_1) \in \mathcal{A}.$$

For any ordered couple $(A,B) \in \mathcal{A}$ and differential operator $\omega \in \mathfrak{D}_{C_R}$ we define

$$\mathfrak{A}_\omega(A,B) = \left\{X | X \in \mathcal{L}, X \subseteq A, \int_A^\omega X = \int_A^\omega B\right\}.$$

It is clear that $\mathfrak{A}_\omega(A,B)$ is a complete sublattice of $\mathcal{L}$.

Let $A \in \mathcal{L}$ and $\int_M^\omega A \neq M$. The submodule A is called an ω – primary if $X \cap Y \subseteq A$ and $X \not\subseteq A$ implies $Y \subseteq \int_M^\omega A$, where $X, Y \in \mathcal{L}$.

Proposition 2. *If A and B are an ω - primary submodules of M and $B \in \mathfrak{A}_\omega(M,A)$, then $A \cap B$ is also an ω – primary submodule of M.*

Proof. Let $X \cap Y \subseteq A \cap B$ and $X \not\subseteq A \cap B$. Then $X \cap Y \subseteq A$ and $X \cap Y \subseteq B$, and also $X \not\subseteq A$ or $X \not\subseteq B$. Using that A and B are an ω – primary it follows $Y \subseteq \int_M^\omega A$ or $Y \subseteq \int_M^\omega B$. Since $\int_M^\omega A = \int_M^\omega B$ from Proposition 1 we have

$$\int_M^\omega A = \int_M^\omega B = \left(\int_M^\omega A\right) \cap \left(\int_M^\omega B\right) = \int_M^\omega (A \cap B).$$

Thus $Y \subseteq \int_M^\omega (A \cap B)$ and $A \cap B$ is an ω – primary module.

In all that follows M will be a differentially noetherian module. A module $A \in \mathcal{L}$ is called irreducible it is not an intersection of two submodules from $\mathcal{L}$ different from A.

The next proposition is analogous to a fact from [5], p. 102.

Proposition 3. *Every module $A \in \mathcal{L}$ is an intersection of a finite number irreducible submodules.*

Proof. We claim that every submodule of M can be written as the intersection of a finite number irreducible submodules. For if not, then since

M is noetherian, it contains a maximal submodule A for which this is not true. Then A must be reducible, say $A = B \cap C$, where $B, C \in \mathfrak{L}$ and $A \subset B$, $A \subset C$. By the maximality of A, each of the submodules B and C is an intersection of a finite number irreducible submodules and so A is also such an intersection. This is a contradiction, and it proves our claim.

It is clear that every irreducible differential submodule of M is an ω – primary, where $\omega \in \mathfrak{D}_{C_R}$. So we have the existence result:

Proposition 4. *Let M will a differentially noetherian module and $\omega \in \mathfrak{D}_{C_R}$. Then every $A \in \mathfrak{L}$ is an intersection of a finite number of ω – primary submodules of M.*

Let $A, B \in \mathfrak{L}$. Then $X \cap B \subseteq A$ and $Y \cap B \subseteq A$ implies $(X+Y) \cap B \subseteq A$, where $X, Y \in \mathfrak{L}$. Since M is a differentially noetherian module there is a maximal module $X \in \mathfrak{L}$ such that $X \cap B \subseteq A$. This module is denoted by $A : B$.

From the definition it is clear that $\left(\bigcap_k A_k\right) : B = \bigcap_k (A_k : B)$.

Proposition 5. *The submodule $A \in \mathfrak{L}$ is an ω – primary if and only if $A : B = A$ for any submodule $B \in \mathfrak{L}$ such that $B \not\subseteq \int_M^\omega A$.*

Proof. Let $A : B = A$ for every module $B \in \mathfrak{L}$ such that $B \not\subseteq \int_M^\omega A$. Suppose that $X \cap Y \subseteq A$ and $X \not\subseteq \int_M^\omega A$. Then $X \not\subseteq A$. So we have $A : X = A$. Hence A is a maximal element in $\mathfrak{L}$ such that $X \cap A \subseteq A$. Thus $Y \subseteq A$ and $Y \subseteq \int_M^\omega A$. Therefore A is an ω – primary module.

Conversely, let A is an ω – primary submodule of M and $X \cap Y \subseteq A$ where $Y \not\subseteq \int_M^\omega A$. Thus $X \subseteq A$ whence $A : Y \subseteq A$. Since $A \subseteq A : Y$ we have $A : Y = A$.

Let $A \in \mathfrak{L}$. By a primary decomposition of A we understand the representation $A = P_1 \cap P_2 \cap \ldots \cap P_n$, where $P_1, \ldots, P_n \in \mathfrak{L}$ are ω – primary modules. Moreover if $\bigcap_{i \neq k} P_i \not\subseteq P_k$, $1 \leq k \leq n$, the primary decomposition of A is called irredundant.

Proposition 6. *Let the module $A \in \mathfrak{L}$ has an irredundant primary decomposition $A = P_1 \cap P_2 \cap \ldots \cap P_n$ such that P_i, $1 \leq i \leq n$, are ω – primary* *modules. If* $P_k \notin \mathfrak{A}_\omega(M, P_i)$ *for some* $1 \le i, k \le n$, *the module* A *is not an* ω *– primary.*

Proof. Let $X = \bigcap_{k \neq i} P_k$. Suppose that A is an ω – primary module. Since $A = X \cap P_i$ and $X \not\subseteq A$ (if $X \subseteq A$ this is a contradiction with the irredundancy of the primary decomposition of A) it follows $P_i \subseteq \int_M^\omega A$. Hence, using the Proposition 3 from [3], we have

$$\int_M^\omega P_i \subseteq \int_M^\omega \left(\int_M^\omega A\right) = \int_M^\omega A = \int_M^\omega \left(\bigcap_k P_k\right) = \bigcap_k \left(\int_M^\omega P_k\right).$$

Thus $\int_M^\omega P_i \subseteq \int_M^\omega P_k$ for every $k = 1, 2, \ldots, n$. Since the index i is an arbitrary, it follows $\int_M^\omega P_i \subseteq \int_M^\omega P_k$ for any indices i and k. Therefore we have $\int_M^\omega P_i = \int_M^\omega P_k$. So $P_k \in \mathfrak{A}_\omega(M, P_i)$ for any i and k, $1 \le i, k \le n$. This is a contradiction, and it establishes the desired conclusion.

We can now state the uniqueness theorem.

Theorem. *Let the module* $A \in \mathfrak{L}$ *has two primary decompositions*

$$A = P_1 \cap P_2 \cap \ldots \cap P_n = Q_1 \cap Q_2 \cap \ldots \cap Q_m,$$

where P_i, $1 \le i \le n$, *and* Q_j, , $1 \le j \le m$, *are* ω *– primary modules,* $P_k \notin \mathfrak{A}_\omega(M, P_i)$, $1 \le i, k \le n$, $i \neq k$ *and* $Q_s \notin \mathfrak{A}_\omega(M, Q_j)$, $1 \le j, s \le m$, $j \neq s$. *Then* $m = n$ *and modules* Q_i *can be renumbered such that* $P_i \in \mathfrak{A}_\omega(M, Q_i)$ *for every* $i = 1, 2, \ldots n$.

Proof. We use induction on n.

For $n = 1$ we have $P_1 = Q_1 \cap Q_2 \cap \ldots \cap Q_m$. Suppose that $m > 1$. This is a contradiction by Proposition 6. Hence $m = 1$ and it is clear that $\int_M^\omega P_1 = \int_M^\omega Q_1$.

Let $n > 1$. Among differential modules $\int_M^\omega P_1, \ldots, \int_M^\omega P_n, \int_M^\omega Q_1, \ldots, \int_M^\omega Q_m$ we choose such one which is not a submodule in some of the others. Without loss of generality we assume that $\int_M^\omega P_1$ is that module. So $\int_M^\omega P_1 \not\subseteq$

$\int_M^\omega P_i$, $i = 2, \ldots, n$ and $\int_M^\omega P_1 \not\subseteq \int_M^\omega Q_j$, $j = 1, 2, \ldots, m$. We will show that there is an index j, $1 \leq j \leq m$, such that $\int_M^\omega P_1 = \int_M^\omega Q_j$. Suppose that $P_1 \notin \mathfrak{A}_\omega(M, Q_j)$ for every $j = 1, 2, \ldots m$.

From the definition of $A : B$ it follows that

$$A : P_1 = (P_1 : P_1) \cap (P_2 : P_1) \cap \ldots \cap (P_n : P_1) =$$

$$= (Q_1 : P_1) \cap (Q_2 : P_1) \cap \ldots \cap (Q_m : P_1).$$

It is clear that $P_1 : P_1 = M$. If suppose $P_1 \subseteq \int_M^\omega P_i$, for $i = 2, \ldots, n$ it follows $\int_M^\omega P_1 \subseteq \int_M^\omega \left(\int_M^\omega P_i \right) = \int_M^\omega P_i$ and this is a contradiction the choise of P_i. Hence $P_1 \not\subseteq \int_M^\omega P_i$ for $i = 2, \ldots, n$. Similarly we prove that $P_1 \not\subseteq \int_M^\omega Q_j$, for $j = 1, 2, \ldots, m$. By Proposition 5 it follows $P_i : P_1 = P_i$, $i = 2, \ldots n$ and $Q_j : P_1 = Q_j$, $j = 1, 2, \ldots m$. Hence

$$P_2 \cap \ldots \cap P_n = Q_1 \cap Q_2 \cap \ldots \cap Q_m.$$

Thus $A = P_2 \cap \ldots \cap P_n$ and this contradict the irredundancy of the decomposition of A.

Therefore exists j, $1 \leq j \leq m$ such that $\int_M^\omega P_1 = \int_M^\omega Q_j$. By renumbering we may write $\int_M^\omega P_1 = \int_M^\omega Q_1$.

Consider $B = P_1 \cap Q_1$. So

$$\int_M^\omega B = \int_M^\omega (P_1 \cap Q_1) = \left(\int_M^\omega P_1 \right) \cap \left(\int_M^\omega Q_1 \right) = \int_M^\omega P_1.$$

By Proposition 2 B is also an ω – primary module. Since $B \subseteq \int_M^\omega B = \int_M^\omega P_1$, by choise of P_1 it follows $B \not\subseteq \int_M^\omega P_i$, $2 \leq i \leq n$ and $B \not\subseteq \int_M^\omega Q_j$, $2 \leq j \leq m$. Now by Proposition 5 we have $P_i : B = P_i$, $2 \leq i \leq n$ and $Q_j : B = Q_j$, $2 \leq j \leq m$. Since $B = P_1 \cap Q_1$ it follows $P_1 : B = Q_1 : B = M$. Using that

$$A : B = (P_1 : B) \cap (P_2 : B) \cap \ldots \cap (P_n : B) =$$

$$= (Q_1 : B) \cap (Q_2 : B) \cap \ldots \cap (Q_m : B)$$

we have $P_2 \cap \ldots \cap P_n = Q_2 \cap \ldots \cap Q_m$. By the induction hypothesis follows $n-1 = m-1$ and $P_i \in \mathfrak{A}_\omega(M, Q_i)$ for every $i = 2, \ldots n$. Since $P_1 \in \mathfrak{A}_\omega(M, Q_1)$ this completes the proof. ■

For lattices $\mathfrak{A}_\omega(A, B)$, $(A, B) \in \mathcal{A}$, we define

$$\mathfrak{A}_\omega(A, B_1) \cap \mathfrak{A}_\omega(A, B_2) = \mathfrak{A}_\omega(A, B_1 \cap B_2),$$

$$\mathfrak{A}_\omega(A, B_1) + \mathfrak{A}_\omega(A, B_2) = \mathfrak{A}_\omega(A, B_1 + B_2),$$

where $(A, B_1), (A, B_2) \in \mathcal{A}$. It is easy to see that relative to these operationns

$$\mathfrak{A}_\omega(A) = \left\{\mathfrak{A}_\omega(A, B) | B \subseteq A\right\}$$

is a complete lattice.

When the ordered couple $(A, B) \in \mathcal{A}$ is from the radical class of the strongly hereditary radical closure I^ω it follows $A = \int_A^\omega B$ and the module A is called B – differentially nil module.

The next result shows the relation between the structure of a differentially noetherian module A and the complete lattice $\mathfrak{A}_\omega(A)$.

Proposition 7. *Let A be a differentially noetherian module and for $\omega \in \mathfrak{D}_{C_R}$ the complete lattice $\mathfrak{A}_\omega(A)$ is a linearly ordered. Then A is either differentially generated by a single element, or B – differentially nil module, where B is differentially generated by a single element.*

Proof. Since the lattice $\mathfrak{A}_\omega(A)$ is a linearly ordered it follows that for B_1 and B_2 where $(A, B_1), (A, B_2) \in \mathcal{A}$ is either $\mathfrak{A}_\omega(A, B_1) \leq \mathfrak{A}_\omega(A, B_2)$, or $\mathfrak{A}_\omega(A, B_2) \leq \mathfrak{A}_\omega(A, B_1)$. Then we have either $B_1 \subseteq \int_A^\omega B_2$, or $B_2 \subseteq \int_A^\omega B_1$. For an arbitrary element $x \in A$ let $\{x\}$ be the differentially generated by x differential submodule of A. Let $x_1 \in A$, $\{x_1\} \neq A$ and $\int_M^\omega \{x_1\} \neq A$. Then there is an element $x_2 \in A$ such that $x_2 \notin \int_M^\omega \{x_1\}$. By linearly ordering in the lattice $\mathfrak{A}_\omega(A)$ it follows $\{x_1\} \subseteq \int_M^\omega \{x_2\}$ whence $\int_M^\omega \{x_1\} \subseteq \int_M^\omega \{x_2\}$. Thus we construct an ascending chain

$$\int_M^\omega \{x_1\} \subseteq \int_M^\omega \{x_2\} \subseteq \cdots \subseteq \int_M^\omega \{x_2\} \subseteq \cdots$$

which becomes stationary for some n. Hence there is an element $x_n \in A$ such that A is $\{x_n\}$ – differentially nil module.

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