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Simultaneous Approximation by Schurer-Stancu Type Operators

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Considering two real parameters α, β which satisfy the inequalities $0 \leq \alpha \leq \beta$ and a given integer $p \geq 0$, in ([4]) were introduced the Schurer-Stancu operators $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1)$, defined for any $f \in C([0,1+p])$ and any $m \in \mathbb{N}$ by

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k} f\left(\frac{k+\alpha}{m+\beta}\right).$$

Some approximation properties of the above operator were there investigated.

In the present paper we study properties of simultaneous approximation for $\widetilde{S}_{m,p}^{(\alpha,\beta)}$. As particular cases, we get similar properties for Schurer, Stancu and Bernstein operators.

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1. Preliminaries

Let α, β be two real parameters satisfying $0 \leq \alpha \leq \beta$ and let $p \geq 0$ be a given integer. In ([4]) were introduced the operators $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1])$

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.1}$$

where $\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$ are the fundamental Schurer polynomials.

The operators (1.1) were called "Schurer-Stancu type operators", because for $\alpha = \beta = 0$ from (1.1) one obtains the Schurer operators (see ([9])) and respectively for $p=0$, from (1.1) one obtains the Stancu operators (see ([11]) and ([12])).

In ([4]) a convergence theorem for the sequence $\left\{\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right\}$ and estimations for the rate of convergence, under different assumptions on the approximated function were established.

Now we study the simultaneous approximation of a function $f \in C^j([0,1+p])$ using the Schurer-Stancu type operators (1.1).

Section 2 provides properties in connection with the derivatives of $\widetilde{S}_{m,p}^{(\alpha,\beta)} f$. As consequences of these properties we get exprimations for $\widetilde{S}_{m,p}^{(\alpha,\beta)} f$ in terms of finite and divided differences of approximated function. In Section 3 we establish a convergence theorem for the sequence $\left\{\left(\frac{d^j}{dx^j}\widetilde{S}_{m,p}^{(\alpha,\beta)}\right) f\right\}_{m\in\mathbb{N}}$, where j is a given non negative integer and $f \in C^j([0,1+p])$.

2. The derivatives of $\widetilde{S}_{m,p}^{(\alpha,\beta)} f$

In what follows we denote by $\widetilde{p}_{m,k}(x)$ the fundamental Schurer polynomials

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{x} x^k (1-x)^{m+p-k}. \tag{2.1}$$

Lemma 2.1. *The polynomials* (2.1) *verify*

$$\widetilde{p}'_{m,k}(x) = (m+p)\left\{\widetilde{p}_{m-1,k-1}(x) - \widetilde{p}_{m-1,k}(x)\right\} = \frac{k-(m+p)x}{x(1-x)}\widetilde{p}_{m,k}(x) \tag{2.2}$$

where $\widetilde{p}'_{m,k}(x)$ *denotes the first order derivative of* $\widetilde{p}_{m,k}(x)$ *and* $\widetilde{p}_{0,0} := 1$, $\widetilde{p}_{k,-1} := 0$, $\widetilde{p}_{m-1,m+p-1} := 0$ $(k \in \mathbb{N}^*)$.

Proof. The assertion follows from (2.1), by direct computation. ∎

Lemma 2.2. *The following equalities*

$$\begin{aligned} \frac{d}{dx}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) &= (m+p)\sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}\Delta_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) \\ &= \frac{m+p}{m+\beta}\sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x)\left[\frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta}; f\right] \end{aligned} \tag{2.3}$$

hold.

P r o o f. From definition (1.1) we have

$$\frac{d}{dx}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right)(x) = \frac{d}{dx}\left(\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)f\left(\frac{k+\alpha}{m+\beta}\right)\right) = \sum_{k=0}^{m+p}\widetilde{p}'_{m,k}(x)f\left(\frac{k+\alpha}{m+\beta}\right).$$

Next, applying Lemma 2.1, we get

$$\frac{d}{dx}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right)(x) = (m+p)\left\{\sum_{k=0}^{m+p}\widetilde{p}_{m-1,k-1}(x)f\left(\frac{k+\alpha}{m+\beta}\right) - \sum_{k=0}^{m+p}\widetilde{p}_{m,k-1}(x)f\left(\frac{k+\alpha}{m+\beta}\right)\right\}.$$

Because $\widetilde{p}_{m-1,0}(x) = 0$, the first sum of the right side of the above equality can be written in the form

$$\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)f\left(\frac{k+\alpha}{m+\beta}\right) = \sum_{k=0}^{m+p-1}\widetilde{p}_{m-1,k}(x)f\left(\frac{k+\alpha+1}{m+\beta}\right).$$

Because $\widetilde{p}_{m,m+p-1}(x) = 0$, the second sum can be written in the form

$$\sum_{k=0}^{m+p}\widetilde{p}_{m,k-1}(x)f\left(\frac{k+\alpha}{m+\beta}\right) = \sum_{k=0}^{m+p-1}\widetilde{p}_{m,k-1}(x)f\left(\frac{k+\alpha}{m+\beta}\right).$$

It follows

$$\begin{aligned}\frac{d}{dx}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right)(x) &= (m+p)\sum_{k=0}^{m+p-1}\widetilde{p}_{m,k-1}(x)\left\{f\left(\frac{k+\alpha+1}{m+p}\right) - f\left(\frac{k+\alpha}{m+\beta}\right)\right\} \\ &= (m+p)\sum_{k=0}^{m+p-1}\widetilde{p}_{m,k-1}(x)\Delta_{\frac{1}{m+\beta}}f\left(\frac{k+\alpha}{m+\beta}\right),\end{aligned}$$

i.e. the first equality (2.2) is proved.

For the second, taking into account of the relation

$$\left[\frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta}; f\right] = (m+\beta)\cdot\Delta_{\frac{1}{m+\beta}}f\left(\frac{k+\alpha}{m+\beta}\right),$$

we get

$$\frac{d}{dx}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \frac{m+p}{m+\beta} \sum_{k=0}^{m+p-1} \widetilde{p}_{m,k-1}(x) \left[\frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta}; f\right]$$

and the proof ends. ∎

Theorem 2.1. *For any non-negative integer $j \le m+p$ the operators (1.1) verify*

$$\frac{d^j}{dx^j}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = (m+p)^{[j]} \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x) \cdot \Delta^j_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) \tag{2.4}$$

where $(m+p)^{[j]} = (m+p)(m+p-1)\ldots(m+p-j+1)$.

P r o o f. For $j=1$ the inequality holds from Lemma 2.2.

Let us to suppose the equality is valid for $j-1$, i.e.

$$\frac{d^{j-1}}{dx^{j-1}}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = (m+p)^{[j-1]} \sum_{k=0}^{m-j+p+1} \widetilde{p}_{m-j+1,k}(x) \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right).$$

Next, applying Lemma 1.1, we get

$$\begin{aligned} \frac{d^j}{dx^j}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) &= (m+p)^{[j-1]} \sum_{k=0}^{m+p-j+1} \widetilde{p}_{m-j+1}(x) \Delta^{j-1}_{\frac{\alpha}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) \\ &= (m+p)^{[j]} \left\{ \sum_{k=0}^{m+p-j+1} \widetilde{p}_{m-j,k-1}(x) \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) \right. \\ &\quad \left. - \sum_{k=0}^{m+p-j+1} \widetilde{p}_{m-j,k}(x) \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) \right\}. \end{aligned} \tag{2.5}$$

Because $\widetilde{p}_{m-j,-1}(x) = 0$, the first sum of the right side of (2.4) can be expressed in the form

$$\sum_{k=0}^{m+p-j+1} \widetilde{p}_{m-j,k-1}(x) \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) = \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x) \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha+1}{m+\beta}\right).$$

Taking into account that $\widetilde{p}_{m-j,m+p-j+1}(x) = 0$, the second sum can be written as follows

$$\sum_{k=0}^{m+p-j+1} \widetilde{p}_{m-j,k}(x) \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) = \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x) \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right).$$

In this way, from (2.5) we get

$$\frac{d^j}{dx^j}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = (m+p)^{[j]} \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x)$$
$$\times\left\{\Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha+1}{m+\beta}\right) - \Delta^{j-1}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right)\right\}$$
$$= (m+p)^{[j]} \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x)\cdot \Delta^{j}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right),$$

i.e. the desired identity is proved by induction after j. ∎

Corollary 2.1. *For any integer j satisfying $1 < j \leq m+p$, the operators (1.1) verify*

$$\text{(2.6)}\quad \frac{d^j}{dx^j}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \frac{(m+p)^{[j]}}{(m+\beta)^j} j! \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x)\left[\frac{k+\alpha}{m+\beta}, \ldots, \frac{k+j+\alpha}{m+\beta}; f\right].$$

Proof. The assertion follows from Theorem 2.1, taking into account of the well known relation between finite and divided differences

$$\left[\frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta}, \ldots, \frac{k+\alpha+j}{m+\beta}; f\right] = \frac{(m+\beta)^j}{j!}\Delta^{j}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right).$$

∎

Corollary 2.2. *The following*

$$\text{(2.7)}\quad \left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{j=0}^{m+p} \binom{m+p}{j} \Delta^{j}_{\frac{1}{m+\beta}} f\left(\frac{\alpha}{m+\beta}\right) x^j$$

holds.

Proof. Applying the Taylor formula, we get

$$\text{(2.8)}\quad \left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{j=0}^{m+p} \frac{d^j}{dx^j}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(0) x^j.$$

From Theorem 2.1, we have

$$\text{(2.9)}\quad \frac{d^j}{dx^j}\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(0) = (m+p)^{[j]} \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x)\Delta^{j}_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right).$$

But $\widetilde{p}_{m-j,0}(x) = 1$ and for any $k \geq 1$, $\widetilde{p}_{m-j,k}(0) = 0$.
It follows that

$$\frac{d^j}{dx^j}\left(\widetilde{S}^{(\alpha,\beta)}_{m,p} f\right)(0) = (m+p)^{[j]} \Delta^j_{\frac{1}{m+\beta}} f\left(\frac{\alpha}{m+\beta}\right). \tag{2.10}$$

In this way, using (2.7), (2.8) and (2.9), we arrive to the desired result (2.6). ∎

Corollary 2.3. *The Schurer operators* $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, $\widetilde{B}_{m,p} := \widetilde{S}^{(0,0)}_{m,p}$, *verify*

$$\frac{d^j}{dx^j}\left(\widetilde{B}_{m,p} f\right)(x) = (m+p)^{[j]} \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x) \Delta^j_{\frac{1}{m}} f\left(\frac{k}{m}\right), \tag{2.11}$$

$$\frac{d^j}{dx^j}\left(\widetilde{B}_{m,p} f\right)(x) = \frac{(m+p)^{[j]}}{m^j} j! \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x) \left[\frac{k}{m+\beta}, \ldots, \frac{k+j}{m+\beta}; f\right], \tag{2.12}$$

$$\left(\widetilde{B}_{m,p} f\right)(x) = \sum_{j=0}^{m+p} \binom{m+p}{j} \Delta^j_{\frac{1}{m}} f(0) x^j. \tag{2.13}$$

Proof. The assertions follow applying Theorem 2.1, Corollary 2.1 and Corollary 2.2 for $\alpha = \beta = 0$. ∎

Corollary 2.4. *The Stancu operator* $P^{(\alpha,\beta)}_m : C([0,1]) \to C([0,1])$, $P^{(\alpha,\beta)}_m := \widetilde{S}^{(\alpha,\beta)}_{m,0}$, *verify the identities*

$$\frac{d^j}{dx^j}\left(P^{(\alpha,\beta)}_m f\right)(x) = m^{[j]} \sum_{k=0}^{m-j} p_{m-j,k}(x) \cdot \Delta^j_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right), \tag{2.14}$$

$$\frac{d^j}{dx^j}\left(P^{(\alpha,\beta)}_m f\right)(x) = \frac{m^{[j]}}{m^j} \cdot j! \sum_{k=0}^{m-j} p_{m-j,k} \left[\frac{k+\alpha}{m+\beta}, \ldots, \frac{k+\alpha+j}{m+\beta}\right], \tag{2.15}$$

$$\left(P^{(\alpha,\beta)}_m f\right)(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta^j_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right), \tag{2.16}$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$ *are the fundamental Bernstein polynomials.*

Proof. The assertions follow applying Theorem 2.1, Corollary 2.1 and Corollary 2.2 for $p = 0$. ∎

The same way, for $\alpha = \beta = 0$ and $p = 0$, we get

Corollary 2.5. *The Bernstein operator* $B_m : C([0,1]) \to C([0,1])$, $B_M := \widetilde{S}_{m,0}^{(0,0)}$, *verify*

$$\frac{d^j}{dx^j}(B_m f)(x) = m^{[j]} \sum_{k=0}^{m-j} p_{m-j}(x) \Delta^j_{\frac{1}{m}} f\left(\frac{k}{m}\right), \tag{2.17}$$

$$\frac{d^j}{dx^j}(B_m f)(x) = \frac{m^{[j]}}{m^j} \cdot j! \sum_{k=0}^{m-j} p_{m-j}(x) \cdot \left[\frac{k}{m}, \dots, \frac{k+j}{m}\right], \tag{2.18}$$

$$(B_m f)(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta^j_{\frac{1}{m}} f(0) x^j. \tag{2.19}$$

3. Simultaneous approximation

The main result of this section is

Theorem 3.1. *For any* $C^j([0,1+p])$ *the sequence* $\left\{\left(\frac{d^j}{dx^j}\widetilde{S}_{m,p}^{(\alpha,\beta)}\right) f\right\}$ *converges to* $f^{(j)}$, *uniformly on* $[0,1]$ $(1 < j \le m+p)$.

Proof. Using the symbol of Landau (see for example ([1])), the relation $\lim_{m\to\infty} \mu\left(\frac{1}{m}\right) = 0$ can be expressed in the form $\mu\left(\frac{1}{m}\right) = o(1)$.

Taking into account the above remark, we can write

$$\begin{aligned}(m+p)^{[j]} \Delta^j_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) \\ = (m+p)^j \{1+o(1)\} \cdot j! \left[\frac{k+\alpha}{m}, \dots, \frac{k+\alpha+j}{m}; f\right]\end{aligned} \tag{3.1}$$

Note that (3.1) follows from Corollary 2.1.

Next, using the mean theorem for divided differences, from (3.1) we get that exists $\xi_k \in \left]\frac{k+\alpha}{m+\beta}, \frac{k+\alpha+j}{m+\beta}\right[$ so that

$$(m+p)^{[j]} \Delta^j_{\frac{1}{m+\beta}} f\left(\frac{k+\alpha}{m+\beta}\right) = \{1+o(1)\} f^{(j)}(\xi_k). \tag{3.2}$$

On the other hand, we have

$$f^{(j)}(\xi_k)=\left\{f^{(j)}(\xi_j)-f^{(j)}\left(\frac{k+\alpha}{m+\beta}\right)\right\}+f\left(\frac{k+\alpha}{m+\beta}\right)=o(1)+f\left(\frac{k+\alpha}{m+\beta}\right),$$

because $\left|\xi_k-\frac{k+\alpha}{m+\beta}\right|<\frac{1}{m+\beta}$ and $f\in C^j([0,1+p])$.

This way follows the identity

$$\{1+o(1)\}f^{(j)}(\xi_k)=\{1+o(1)\}+f^{(j)}\left\{\frac{k+\alpha}{m+p}\right\}$$

$$(3.3)\qquad = o(1)+(1+o(1))f^{(j)}\left(\frac{k+\alpha}{m+\beta}\right)=f^{(j)}\left(\frac{k+\alpha}{m+\beta}\right)+o(1)\,.$$

From Theorem 2.1 taking into account that $\xi_k\in\left]\frac{k+\alpha}{m+\beta};\frac{k+\alpha+j}{m+\beta}\right[$, we get

$$\left(\frac{d^j}{dx^j}\widetilde{S}_{m,p}^{(\alpha,\beta)}\right)f = \sum_{k=0}^{m+p-j}\widetilde{p}_{m-j,k}(x)\left\{f^{(j)}\left(\frac{k+\alpha}{m+\beta}\right)+o(1)\right\}$$

$$(3.4)\qquad = \left(\widetilde{S}_{m-j}^{(\alpha,\beta)}f^{(j)}\right)(x)+o(1).$$

Because the sequence $\left(\widetilde{S}_{m-j}g\right)_{m\geq j}$ converges to g, uniformly on $[0,1]$ for any $g\in C([0,1+p])$ (see ([4])), from (3.4) we arrive to the desired result. ■

Corollary 3.1. *Let* $\widetilde{B}_{m,p}:=\widetilde{S}_{m,p}^{(0,0)}$ *be the Schurer operator. The sequence* $\left\{\left(\frac{d^j}{dx^j}\widetilde{B}_{m,p}\right)f\right\}_{m\in\mathbb{N}}$ *converges to* $f^{(j)}$*, uniformly on* $[0,1]$*, for any* $f\in C^{(j)}([0,1+p])$.

Proof. We apply Theorem 3.1 for $\alpha=\beta=0$. ■

Corollary 3.2. *Let* $P_m^{(\alpha,\beta)}:=\widetilde{S}_{m,0}^{(\alpha,\beta)}$ *be the Stancu operator. The sequence* $\left\{\left(\frac{d^j}{dx^j}P_m^{(\alpha,\beta)}\right)f\right\}_{m\geq 0}$ *converges to* $f^{(j)}$*, uniformly on* $[0,1]$*, for any* $f\in C^{(j)}([0,1])$.

Proof. We take $p=0$ in Theorem 3.1. ■

Corollary 3.3. *Let* $B_m:=\widetilde{S}_{m,0}^{(0,0)}$ *be the Bernstein operator. The sequence* $\left\{\left(\frac{d^j}{dx^j}B_m\right)f\right\}_{m\geq 0}$ *converges to* $f^{(i)}$*, uniformly on* $[0,1]$*, for any* $f\in C^{(j)}([0,1])$.

Proof. Assertion follows from Theorem 3.1, for $\alpha = \beta = 0$ and $p = 0$. ∎

Finally, let us to dedicate the paper to our dear Professor, Academician Dimitrie D. Stancu, with the compliments of the author.

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