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Variational Methods for Certain Analytic Stieltjes Integrals with Probability Measures on $[a, b]$ and Applications

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We derive variational formulas for the functions stated in the title and apply them to the Nevanlinna univalent functions of the class N_2.

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1. Introduction.

Let $M[a, b]$ denote the set of all probability measures $\mu(t)$ on $I[a, b] \equiv \{t : a \leq t \leq b\}$, where $-\infty < a < b < +\infty$, and let $V_g[a, b]$ denote certain normal and compact class of analytic functions of the form

$$f(z) = \int_a^b g(z, t)\, d\mu(t), \quad z \in \Delta \equiv \{z : |z| < 1\}, \tag{1}$$

where $g(z, t)$ is a given analytic function in the product $\Delta \times I[a, b]$.

In this paper we will derive variational methods which in comparison with earlier variational methods ([1], [2, pp. 504–519] and [3]) yield more precise information for the extremal functions of a given bounded real-valued continuous functional in $V_g[a, b]$.

2. Variational formulas for the class $V_g[a,b]$.

The following vatiational methods and results represented by Theorems 1 and 2 are new.

Theorem 1. *Let ε with $-1<\varepsilon<1$, $\varepsilon\neq 0$, be an arbitrary number and let the function $f(z)$ belong to the class $V_g[a,b]$. Then the varied function*

$$(2)\qquad f_*(z)=\int_a^b g\left(z,\frac{[b-a-\varepsilon(a+b)]t+2\varepsilon ab}{b-a+\varepsilon(a+b)-2\varepsilon t}\right)d\mu(t),\quad z\in\Delta,$$

also belongs to the class $V_g[a,b]$ and it has the following asymptotic representation

$$(3)\qquad f_*(z)=f(z)-\frac{2\varepsilon}{b-a}\int_a^b(t-a)(b-t)g_t'(z,t)\,d\mu(t)+O(\varepsilon^2),\qquad z\in\Delta,$$

where $O(\varepsilon^2)$ denotes a magnitude, the ratio of which to ε^2 is uniformly bounded for z lying in an arbitrary closed set of the disk Δ.

P r o o f. The linear fractional function

$$(4)\qquad \tau=\frac{[b-a-\varepsilon(a+b)]t+2\varepsilon ab}{b-a+\varepsilon(a+b)-2\varepsilon t},\quad t\in I[a,b],\quad -1<\varepsilon<1,\quad \varepsilon\neq 0,$$

for fixed ε, increases with t from a to b. This property of (4) permits us to substitute the right-hand side of (4) for t in the integrand of (1) to obtain (2). The function (2) belongs to the class $V_g[a,b]$ with the probability measure

$$\nu(\tau):=\mu\left(\frac{[b-a+\varepsilon(a+b)]\tau-2\varepsilon ab}{b-a-\varepsilon(a+b)+2\varepsilon\tau}\right)\in M[a,b],\ \tau\in I[a,b].$$

For $z\in\Delta$ and sufficiently small $|\varepsilon|$, it follows from (4) that the Taylor series

$$(5)\qquad g(z,\tau)=g(z,t)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial^n g(z,t)}{\partial t^n}(\tau-t)^n$$

$$=g(z,t)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{2^n\varepsilon^n(t-a)^n(t-b)^n}{[b-a-\varepsilon(2t-a-b)]^n}\frac{\partial^n g(z,t)}{\partial t^n}$$

$$=g(z,t)+\sum_{n=1}^{\infty}\frac{2^n\varepsilon^n(t-a)^n(t-b)^n}{n!(b-a)^n}\frac{\partial^n g(z,t)}{\partial t^n}\sum_{\nu=0}^{\infty}\binom{n+\nu-1}{\nu}\varepsilon^\nu\left(\frac{2t-a-b}{b-a}\right)^\nu.$$

Thus from (5), (4), (2) and (1) we obtain (3), which complete the proof of Theorem 1. ∎

Theorem 2. *For a given point z of the disk Δ and a given analytic function $\Phi(u_0, u_1, \dots, u_n; z)$, $n \geq 0$, on the set $\cup_{V_g[a,b]}\{f(z), f'(z), \dots, f^{(n)}(z); z\}$, the minimum (the maximum) of the functional*

(6) $$\Re\Phi(f(z), f'(z), \dots, f^{(n)}(z); z)$$

in the class $V_g[a,b]$ is attained only either in the subclass $V_g^1[a,b] \subset V_g[a,b]$ of functions

(7) $$f(z) = cg(z,a) + (1-c)g(z,b) \in V_g^1[a,b], \quad 0 \leq c \leq 1,$$

or in the subclass $V_g^2[a,b] \subset V_g[a,b]$ of functions

(8) $$f(z) = \sum_{k=1}^{p} c_k g(z, t_k) \in V_g^2[a,b]$$

for some integer $p \geq 1$ and

(9) $$a \leq t_1 \leq t_2 \leq \ldots \leq t_p \leq b, \quad 0 \leq c_k \leq 1, \quad \sum_{k=1}^{p} c_k = 1,$$

where $t_1, t_2, \dots, t_p$ are between the numbers a and b and the roots are in the interval $a \leq t \leq b$ of the equation

(10) $$P(t) \equiv \Re\left[\sum_{s=0}^{n} \frac{\partial\Phi[f(z)]}{\partial u_s} \frac{\partial^{s+1} g(z,t)}{\partial t \partial z^s}\right] = 0$$

in t for which we assume that at the extremum of (6) it is not an identity in $t \in I[a,b]$ where $\Phi[f(z)] \cong \Phi(f(z), f'(z), \dots, f^{(n)}(z); z)$.

Proof. The extremal functions $f(z) \in V_g[a,b]$ exist since the functional (6) is continuous and bounded on $V_g[a,b]$ and the class $V_g[a,b]$ is normal and compact in Δ. If we set

(11) $$u_s = f^{(s)}(z), \quad u_s^* = f_*^{(s)}(z) \quad (0 \leq s \leq n),$$

then the increments by the aymptotic formula (3) are

(12) $$du_s = u_s^* - u_s = -\frac{2\varepsilon}{b-a}\int_a^b (t-a)(b-t)\frac{\partial^{s+1} g(z,t)}{\partial t \partial z^s}\, d\mu(t) + O(\varepsilon^2) \ (0 \leq s \leq n).$$

Further we introduce the abridget notations

(13) $$\Phi \equiv \Phi(u_0, u_1, \dots, u_n; z), \quad \Phi^* \equiv \Phi(u_0^*, u_1^*, \dots, u_n^*; z),$$

where u_s and u_s^* ($0 \le s \le n$) are given by (11). Then for sufficiently small $|\varepsilon|$, we have the Taylor series

$$\Phi^* = \Phi + \sum_{\nu=1}^{\infty} \frac{1}{\nu!}\left(\sum_{s=0}^{n} \frac{\partial}{\partial u_s} du_s\right)^{\nu} \Phi \tag{14}$$

for the functions (13). From (14) and (12) we obtain

$$\Phi^* = \Phi - \frac{2\varepsilon}{b-a}\sum_{s=0}^{n} \frac{\partial \Phi}{\partial u_s}\int_a^b (t-a)(b-t)\frac{\partial^{s+1} g(z,t)}{\partial t \partial z^s}\, d\mu(t) + O(\varepsilon^2). \tag{15}$$

It follows from (15) that the varied functional

$$\Re\Phi^* = \Re\Phi - \frac{2\varepsilon}{b-a}\int_a^b (t-a)(b-t)\Re\left[\sum_{s=0}^{n} \frac{\partial \Phi}{\partial u_s}\frac{\partial^{s+1} g(z,t)}{\partial t \partial z^s}\right] d\mu(t) + O(\varepsilon^2). \tag{16}$$

The extremality of the function $f(z)$ in the class $V_g[a,b]$ and the arbitrariness of ε imply that the coefficient of ε in (16) vanishes, i.e. that

$$\int_a^b (t-a)(b-t)\Re\left[\sum_{s=0}^{n} \frac{\partial \Phi}{\partial u_s}\frac{\partial^{s+1} g(z,t)}{\partial t \partial z^s}\right] d\mu(t) = 0. \tag{17}$$

If the conditions for the equation (10) hold, then the equation of the extremality (17) is fulfilled if and only if the measure $\mu(t)$ is a step function with points of discontinuity at a and b and the roots of the equation (10) with respect to t in the closed interval $I[a,b]$ and the corresponding jumps with a sum equal to unit. In fact, this is evident if $\mu(t)$ is a corresponding step function. Conversely, it follows from Goluzin variational formula applied to the class $V_g[a,b]$ (see [1], [2, p. 506, formula (7)] and [3, p. 93, formula (19)]) that $\mu(t)$ is a constant between any two adjacent roots of the equation (10) for the extremal function $f(z)$ (see the comments for our formulas (27)–(28) in [3, pp. 94–95]). Hence the extremal functions $f(z)$ belong to the subclasses $V_g^1[a,b] \subset V_g[a,b]$ and $V_g^2[a,b] \subset V_g[a,b]$ of functions (7) and (8)–(9), respectively.

This completes the proof of Theorem 2. ∎

The following Theorems 3 and 4 develop the ideas in [3].

Theorem 3, *If the conditions for the equation (10) hold, then the function*

$$Q(t) \equiv \Re\left[\sum_{s=0}^{n} \frac{\partial \Phi[f(z)]}{\partial u_s} \cdot \frac{\partial^s g(z,t)}{\partial z^s}\right] \tag{18}$$

in t has equal values at all points of the discontinuity of the measure $\mu(t)$ for the extremal functions (7) and (8)–(9) of the functional (6).

P r o o f. It follows from the conditions for the equation (10) that we have

$$\frac{\partial\Phi[f(z)]}{\partial u_s} \equiv \frac{\partial}{\partial u_s}\Phi(f(z), f'(z), \dots, f^{(n)}(z); z) \neq 0 \tag{19}$$

at least for one $s \in \{0, 1, \dots, n\}$. Further, let the real number ε be with sufficiently small $|\varepsilon|$. If the extremal function $f(z) \in V_g^2[a,b]$ and in (8)–(9) we substitute $c_k + \varepsilon$ and $c_{k+1} - \varepsilon$ for c_k and c_{k+1}, respectively, then the varied function

$$f_{**}(z) = f(z) + \varepsilon[g(z, t_k) - g(z, t_{k+1})] \tag{20}$$

also belongs to the subclass $V_g^2[a,b]$. If we analogously set

$$u_s = f^{(s)}(z), \quad u_s^{**} = f_{**}^{(s)}(z) \quad (0 \le s \le n), \tag{21}$$

then the increments by formula (20) are

$$du_s = u_s^{**} - u_s = \varepsilon\left[\frac{\partial^s g(z,t_k)}{\partial z^s} - \frac{\partial^s g(z,t_{k+1})}{\partial z^s}\right] \quad (0 \le s \le n). \tag{22}$$

For brevity, we again denote

$$\Phi \equiv \Phi(u_0, u_1, \dots, u_n; z), \quad \Phi^{**} \equiv \Phi(u_0^{**}, u_1^{**}, \dots, u_n^{**}; z), \tag{23}$$

where u_s and u_s^{**} $(0 \le s \le n)$ are given by (21). Then the corresponding Taylor series (14) for the functions (23) and (22) analogously yield

$$\Re\Phi^{**} = \Re\Phi + \varepsilon\Re\left\{\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\left[\frac{\partial^s g(z,t_k)}{\partial z^s} - \frac{\partial^s g(z,t_{k+1})}{\partial z^s}\right]\right\} + O(\varepsilon^2). \tag{24}$$

The extremality of $f(z)$ in (24), the arbitrariness of ε and the inequality (19) imply the condition

$$\Re\left\{\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\left[\frac{\partial^s g(z,t_k)}{\partial z^s} - \frac{\partial^s g(z,t_{k+1})}{\partial z^s}\right]\right\} = 0. \tag{25}$$

The condition (25) shows that the function (18) has equal values at any two adjacent points of discontinuity t_k and t_{k+1} of the considered measure $\mu(t)$. If the extremal function $f(z) \in V_g^1[a,b]$, the assertion is established in the same way.

This completes the proof of Theorem 3. ■

Theorem 4. *If the conditions for the equation (10) hold, then the equation (10) in t has at least one root inside the intervals between any two adjacent points of discontinuity of the measure* $\mu(t)$ *for the extremal functions* $f(z) \in V_g^{1,2}[a,b]$ *of the functional (6).*

Proof. According to the Rolle theorem applied to Theorem 3 the derivative $Q'(t)$ vanishes between any two adjacent points of discontinuity of the considered measure $\mu(t)$. In addition, we have

$$Q'(t) = \frac{\partial}{\partial t}\Re\left[\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\frac{\partial^s g(z,t)}{\partial z^s}\right] = \Re\left[\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\frac{\partial^{s+1} g(z,t)}{\partial t\partial z^s}\right] = P(t), \tag{26}$$

having in mind (10). Now Theorem 4 follows from (26). ■

Theorems 3 and 4 determine upper bound of the number p in Theorem 2. In particular, if the equation (10) is reduced to an algebraic equation in t, then Theorem 3 and 4 reduce by half the upper bound of the number of points of discontinuity of $\mu(t)$ determined by its roots and the points a and b (see, for example, in [3, pp. 95–96]).

3. Applications of Theorems 2, 3 and 4 to the Nevanlinna univalent functions of the class N_2.

The class N_2 is obtained from (1) for $a = -1$, $b = 1$ and

$$g(z,t) = \frac{z}{1-tz}, \quad z \in \Delta, \quad t \in I[-1,1] \tag{27}$$

(see [3]), i.e. $N_2 \equiv V_g[-1,1]$, where g is given by (27).

Theorem 5. *For a given point z of the disk* Δ *and for each function* $f(z) \in N_2$*, we have the following sharp estimates*

$$\left|\frac{z}{1+z}\right| \le |f(z)| \le \left|\frac{z}{1-z}\right|, \quad \left|z-\frac{1}{2}\right| \le \frac{1}{2}, \tag{28}$$

$$\left|\frac{z}{1-z}\right| \le |f(z)| \le \left|\frac{z}{1+z}\right|, \quad \left|z+\frac{1}{2}\right| \le \frac{1}{2}, \tag{29}$$

$$\frac{|\Im z|}{|1-z^2|} \le |f(z)| \le \frac{1}{|\Im \frac{1}{z}|}, \quad \left|z \pm \frac{1}{2}\right| \ge \frac{1}{2}, \tag{30}$$

where for $z \neq 0$*, the equalities hold true only:*

(i) for the functions $f(z) \equiv g(z,-1)$ *and* $f(z) \equiv g(z,1)$*, determined by (27), on the left-hand side and the right-hand side of (28), respectively;*

(ii) for the functions $f(z) \equiv g(z,1)$ *and* $f(z) \equiv g(z,-1)$*, determined by (27), on the left-hand side and the right-hand side of (29), respectively;*

(iii) for the functions (7), formed by (27) at $t = \mp 1$*, with* $c = (1/2)(1 + \Re(1/z))$*, and (27) with* $t = \Re(1/z)$ *on the lefi-hand side and the right-hand side of (30), respectively.*

P r o o f. We apply Theorem 2 for $n = 0$ and the function

$$\Phi(u_0; z) = \ln \frac{u_0}{z}, \quad z \in \Delta, \tag{31}$$

on the set $\cup_{N_2}\{f(z); z\}$. The functional (6) for (31) is

$$\Re\Phi(f(z); z) = \Re \ln \frac{f(z)}{z} = \ln \left|\frac{f(z)}{z}\right|, \quad f(z) \in N_2, \ z \in \Delta. \tag{32}$$

The equation (10) for the functions (31) and (27) is

$$P(t) \equiv \Re \left[\frac{z}{f(z)} \frac{z}{(1-tz)^2}\right] = 0, \ f(z) \in N_2, \ z \in \Delta. \tag{33}$$

For $z = 0$, the equation (33) is identically fulfilled for all values of $t \in I[-1,1]$ according to (1) and (27). For $z = 0$ all inequalities (28)–(30) are reduced to equalities for all functions $f(z) \in N_2$. For fixed $z \in \Delta$ with $z \neq 0$, the extremal functions $f(z) \in N_2$ for the extrema of functional (32) and the functional $|f(z)|$ in the class N_2 are one and the same. For fixed $z \in \Delta$ with $z \neq 0$, the equation (33) is quadratic in t, and hence, according to Theorems 2, 3 and 4, these extremal functions are of the forms

$$f(z) = c\frac{z}{1+z} + (1-c)\frac{z}{1-z} \in N_2, \quad 0 \le c \le 1, \tag{34}$$

$$f(z) = c\frac{z}{1+z} + (1-c)\frac{z}{1-tz} \in N_2, \quad 0 \le c \le 1, \quad -1 \le t \le 1, \tag{35}$$

$$f(z) = c\frac{z}{1-tz} + (1-c)\frac{z}{1-z} \in N_2, \quad 0 \le c \le 1, \quad -1 \le t \le 1, \tag{36}$$

$$f(z) = \frac{z}{1-tz} \in N_2, \quad -1 \le t \le 1. \tag{37}$$

The functions (35) can be written in the form

$$f(z)=\frac{z(1-\tau z)}{(1+z)(1-tz)},\ \tau=c(t+1)-1,\ -1\le\tau\le t\le 1. \tag{38}$$

For $z\neq 0$ from (38) we obtain

$$|f(z)|^2=\frac{A(z,\tau)}{A(z,-1)A(z,t)} \tag{39}$$

where we have introduced the positive function

$$A(z,x)=x^2-2x\Re\frac{1}{z}+\frac{1}{|z|^2},\ z\neq 0,\ -\infty<x<+\infty. \tag{40}$$

The function (40) decreases for $-\infty<x\le\Re(1/z)$ and increases for $\Re(1/z)\le x<+\infty$. Hence:

(a) For $\Re(1/z)\ge 1$, we have

$$A(z,-1)\ge A(z,\tau)\ge A(z,t)\ge A(z,1). \tag{41}$$

From (41) and (39) we obtain the inequalities (28) for the functions (35) (or (38)) with equalities only for $c=1$ and $c=0$, $t=1$ on the left-hand side and the right-hand side respectively;

(b) For $\Re(1/z)\le -1$, we have

$$A(z,-1)\le A(z,\tau)\le A(z,t)\le A(z,1). \tag{42}$$

From (42) and (39) we obtain the inequalities (29) for the functions (35) (or (38)) with equalities only for $c=0$, $t=1$ and $c=1$ on the left-hand side and the right-hand side, respectively;

(c) For $-1\le\Re(1/z)\le 1$, we have the following three possibilities

$$A(z,-1)\ge A(z,\tau)\ge A(z,t)\ge A(z,\Re\frac{1}{z}), \tag{43}$$

$$A(z,-1)\ge A(z,\tau)\ge A(z,\Re\frac{1}{z}),\quad A(z,\Re\frac{1}{z})\le A(z,t)\le A(z,1), \tag{44}$$

$$A(z,-1)\ge A(z,\Re\frac{1}{z}),\quad A(z,\Re\frac{1}{z})\le A(z,\tau)\le A(z,t)\le A(z,1). \tag{45}$$

From (43)–(45) and (39) we obtain the inequalities (30) for the functions (38) (or (35)) with equalities only for $\tau=\Re(1/z)$; $t=1$ and $\tau=-1$, $t=\Re(1/z)$ on the left-hand side and the right-hand side, respectively.

If in the function (35) we replace z, t and c by $-z$, $-t$ and $1-c$, respectively, then we will obtain the function (36), multiplied by -1, for which the inequalities (28)–(30) hold as well. Hence the inequalities (28)–(30) hold in the full class N_2.

This completes the proof of Theorem 5. ■

Remark 1. All extremal functions for (28), (29) and the left-hand side of (30) can be obtained from the functions (34) for appropriate values of c ($c=0$, $c=1$ and $c=(1/2)(1+\Re(1/z))$, respectively). The extremal function for the right-hand side of (30) is obtained from (37) for $t=\Re(1/z)$, where it cannot be obtained from (34) for $-1<t<1$. This example shows that the existence of the second extremal subclass $V_g^2[a,b]$ in the general Theorem 2 cannot be neglected.

Theorem 6. *For a given point z of the disk Δ and for each function $f(z)\in N_2$, we have the following sharp inequalities*

$$arg\,\frac{1}{1\pm z}\le\ arg\,\frac{f(z)}{z}\le\ arg\,\frac{1}{1\mp z}, \tag{46}$$

where for $\Im z\neq 0$, the equalities hold true only:

(i) if $\Im z>0$, for the functions $f(z)\equiv g(z,-1)$ and $f(z)\equiv g(z,1)$, determined by (27), on the left-hand side and the right-hand side of (46), respectively;

(ii) if $\Im z<0$, for the functions $f(z)\equiv g(z,1)$ and $f(z)\equiv g(z,-1)$, determined by (27), on the left-hand side and the right-hand side of (46), respectively.

Proof. We apply again Theorem 2 for $n=0$ but for the function

$$\Phi(u_0;z)=-i\ln\frac{u_0}{z},\quad z\in\Delta, \tag{47}$$

on the set $\cup_{N_2}\{f(z);z\}$. The functional (6) for (47) is

$$\Re\Phi(f(z);z)=\Re\left[-i\ln\frac{f(z)}{z}\right]=\ \arg\,\frac{f(z)}{z},\quad f(z)\in N_2,\quad z\in\Delta, \tag{48}$$

where "arg" everywhere is in the interval $(-\pi,\pi]$. The equation (10) for the functions (47) and (27) is

$$P(t)\equiv\Re\left[-i\frac{z}{f(z)}\frac{z}{(1-tz)^2}\right]=0,\ f(z)\in N_2,\ z\in\Delta. \tag{49}$$

For fixed $z\in\Delta$ with $\Im z=0$, the equation (49) is identically fulfilled for all values of $t\in I[-1,1]$ according to (1) and (27). For $\Im z=0$ ($z\in\Delta$) all inequalities (46) are reduced to equalities for all functions $f(z) \in N_2$. For fixed $z \in \Delta$ with $\Im z \neq 0$, the equation (49) is quadratic in t, and hence, according to Theorems 2, 3 and 4, the extremal functions of the functional (48) are also of the form (34)–(37). We examine again the function (35) (or (38)).

(a) For $\Im z > 0$, it is geometrically clear that

$$\arg(1-z) \underset{t=1}{\leq} \arg(1-tz) \underset{\tau=t}{\leq} \arg(1-\tau z) \underset{\tau=-1}{\leq} \arg(1+z) \tag{50}$$

where τ and t are those in (38). From (38) and (50) we obtain

$$\begin{aligned}\arg\frac{f(z)}{z} &= [\arg(1-\tau z) - \arg(1+z)] - \arg(1-tz)\\ &\underset{\tau=-1}{\leq} 0 - \arg(1-tz) \underset{t=1}{\leq} \arg\frac{1}{1-z}\end{aligned} \tag{51}$$

and

$$\begin{aligned}\arg\frac{f(z)}{z} &= [\arg(1-\tau t) - \arg(1-tz)] - \arg(1+z)\\ &\underset{\tau=t}{\geq} 0 - \arg(1+z) = \arg\frac{1}{1+z}.\end{aligned} \tag{52}$$

(b) For $\Im z < 0$, it is geometrically clear that

$$\arg(1+z) \underset{\tau=-1}{\leq} \arg(1-\tau z) \underset{\tau=t}{\leq} \arg(1-tz) \underset{t=1}{\leq} \arg(1-z), \tag{53}$$

where τ and t are those in (38). From (38) and (53) we obtain

$$\begin{aligned}\arg\frac{f(z)}{z} &= [\arg(1-\tau z) - \arg(1-tz)] - \arg(1+z)\\ &\underset{\tau=t}{\leq} 0 - \arg(1+z) = \arg\frac{1}{1+z}\end{aligned} \tag{54}$$

and

$$\begin{aligned}\arg\frac{f(z)}{z} &= [\arg(1-\tau z) - \arg(1+z)] - \arg(1-tz)\\ &\underset{\tau=-1}{\geq} 0 - \arg(1-tz) \geq \arg\frac{1}{1-z}.\end{aligned} \tag{55}$$

Thus from (51)–(52) and (54)–(55) we obtain the inequalities (46) for the functions (35) (or (38)) with the equalities indicated in section (i) and (ii) of

Theorem 6. Further the argumentation is like the one in the end of the proof of Theorem 5.

This completes the proof of Theorem 6. ■

R e m a r k 2. All extremal functions for (46) can be obtained from the functions (34) for appropriate values of c ($c = 0$ and $c = 1$, respectively) as well as from the functions (37) for $t = \pm 1$. This example shows that the first extremal subclass $V_g^1[a, b]$ as well as the second extremal subclass $V_g^2[a, b]$ in the general Theorem 2 can contain all the extremal functions.

Theorems 5 and 6 have been obtained in [4]–[6] by several different methods.

References

[1] G. M. G o l u z i n, On a variational methods in the theory of analytic functions, *Amer. Math. Soc. Transl.*, (2) **18** (1961), 1–14.

[2] G. M. G o l u z i n, *Geometrical Theory of Functions of a Complex Variable*, (Second Edition), Izdat. "Nauka", Moscow, 1966, 628 pp. ; *Amer. Math. Soc. Transl. of Math. Monographs*, **26** (1969), 678.

[3] P. G. T o d o r o v, Sharp estimates for the coefficients of the inverse functions of the Nevanlinna univalent functions of the classes N_1 and N_2, *Abh.Math. Sem. Univ. Hamburg*, **68** (1998), 91–102.

[4] M. O. R e a d e and P. G. T o d o r o v, On certain analytic (Nevanlinna) functions, *Michigan Math. J.*, **32** (1985), 59–64.

[5] M. O. R e a d e and P. G. T o d o r o v, Extremal problems for certain analytic (Nevanlinna) functions, *Rendiconti del Circolo Matematico di Palermo*, Serie II **XXXIV** (1985), 275–282.

[6] M. O. R e a d e and P. G. T o d o r o v, Extremal problems for certain Nevanlinna analytic functions, *Serdica, Bulgaricae mathematicae publicationes*, **12**, (1986), 390–398.

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