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Some Properties of Rational Functions with Prescribed Poles and Restricted Zeros

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Let $P(z)$ be a polynomial of degree not exceeding n and $W(z) = \prod_{j=1}^{n}(z - a_j)$, where $|a_j| \geq 1$, $j = 1, 2, \ldots, n$. If the rational function $r(z) = P(z)/W(z)$ does not vanish in $|z| > K$, then for $K = 1$, it is known that

$$|r'(z)| \geq \left\{\frac{1}{2}|B'(z)| - \frac{1}{2}(n - m)\right\} |r(z)|$$

where m is the number of zeros of $r(z)$. In this paper we consider the case when $K > 1$ and obtain a sharp result. We also prove a generalization of a result due to Xin Li, Mohapatra and Rodriguez, which also extends some polynomial inequalities to a class of rational functions.

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1. Introduction and statement of results.

Let $\mathbf{P}_n$ denote the class of all complex polynomials of degree at most n. Let D_{k-} denote the region inside the circle $T_k = \{z : |z| = k > 0\}$ and D_{k+} the region outside T_k. For $a_j \in C$ with $j = 1, 2, \ldots, n$, we write

$$W(z) = \prod_{j=1}^{n}(z - a_j), \qquad B(z) = \prod_{j=1}^{n}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right)$$

and

$$R_n = R_n(a_1, a_2, \ldots, a_n) = \left\{\frac{P(z)}{w(z)}, P \in \mathbf{P}_n\right\}.$$

Then R_n is the set of all rational functions with at most n poles $a_1, a_2, \ldots$, a_n and with a finite limit at infinity. We shall always assume that these poles lie in D_{1+}. Also we observe that $B(z) \in R_n$. For f defined on T_k in the complex plane, we set

$$M(f, k) = \sup_{z \in T_k} |f(z)|.$$

Let $P \in \mathbf{P}_n$ then concerning the estimate of $M(P', 1)$ on T_1, we have the following famous result due to Bernstein [9].

Theorem A. *Let $P \in \mathbf{P}_n$, then*

$$M(P', 1) \leq nM(P, 1). \tag{1}$$

In the literature [4, 8], there exist several improvements and generalizations of Theorem A.

Recently Li, Mohapatra and Rodriguez [6] obtained Berstein-type inequalities for rational functions $r(z) \in R_n$ with prescribed poles $a_1, a_2, \ldots, a_n$ replacing z^n by Blaschke product $B(z)$. Among other things they proved the following results for rational functions with restricted zeros.

Theorem B. *Suppose that $r \in R_n$ and all the zeros of r lie in $T_1 \cup D_{1+}$. Then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}|B'(z)|M(r, 1). \tag{2}$$

Theorem C. *Suppose that $r \in R_n$, where r has exactly n poles at a_1, $a_2, \ldots, a_n$ and all the zeros of r lie in $T_1 \cup D_{1-}$, then for $z \in T_1$,*

$$|r'(z)| \geq \left\{\frac{1}{2}|B'(z)| - \frac{1}{2}(n - m)\right\}|r(z)|, \tag{3}$$

where m is the number of zeros of r.

As an improvement in (2) authors [2] proved the following:

Theorem D. *Let $r \in R_n$ and all the zeros of r lie in $T_1 \cup D_{1+}$. If t_1, $t_2, \ldots, t_n$ are the zeros of $B(z) + \lambda$, and $s_1, s_2, \ldots, s_n$ are the zeros of $B(z) - \lambda$, where $\lambda \in T_1$, then for $z \in T_1$*

$$|r'(z)| \leq \frac{1}{2}|B'(z)|\left\{\left(\max_{1 \leq i \leq n}|r(t_i)|\right)^2 + \left(\max_{1 \leq i \leq n}|r(s_i)|\right)^2\right\}^{1/2}. \tag{4}$$

Aziz and Zargar [3] considered a class of rational functions R_n not vanishing in $T_k \cup D_{k-}$, where $k \geq 1$ and proved the following generalization of Theorem B.

Theorem E. *Suppose $r \in R_n$ and all the zeros of r lie in $T_k \cup D_{k+}$, where $k \geq 1$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k-1)}{k+1}\frac{|r(z)|^2}{\{M(r,1)\}^2}\right\} M(r,1). \tag{5}$$

It is natural to ask what results in (3) and (4), if we consider the class of rational functions R_n analogous to Theorem E? In reply to this, we first consider the class of rational functions R_n, not vanishing in $T_k \cup D_{k+}$, where $k \leq 1$ and prove the following generalization of Theorem C.

Theorem 1. *Suppose $r \in R_n$, where r has exactly n poles at $a_1, a_2, \ldots,$ a_n and all the zeros of r lie in $T_k \cup D_{k-}$, $k \leq 1$ then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{2m - n(1+k)}{1+k}\right\}|r(z)|, \tag{6}$$

where m is the number of zeros of r. The result is the best possible and equality holds for

$$r(z) = \frac{(z+k)^m}{(z-a)^n} \quad \textit{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n \quad \textit{at} \quad z = 1, a \geq 1.$$

As an immediate consequence of Theorem 1, we have the following interesting generalization of inequality (12) in [6, p. 526], where r has exactly n zeros in $T_k \cup D_{k-}$.

Corollary 1. *Suppose $r \in R_n$ and all the zeros of r lie in $T_k \cup D_{k-}$, where $k \leq 1$, then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{n(1-k)}{1+k}\right\}|r(z)|. \tag{7}$$

The result is sharp and equality holds for

$$r(z) = \left(\frac{z+k}{z-a}\right)^n \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n \quad \text{at} \quad z = 1, a \geq 1.$$

Inequality (7) is a generalization of a polynomial inequality due to Malik [7] and for $k = 1$, it generalizes the polynomial inequality due to Turan [10].

While seeking the desired extension of Theorem D analogous to Theorem E, we have been able to prove the following.

Theorem 2. *Let $r \in R_n$ and all zeros of r lie in $T_k \cup D_{k+}$. If $t_1, t_2, \ldots, t_n$ are the zeros of $B(z) + \lambda$, and $s_1, s_2, \ldots, s_n$ are the zeros of $B(z) - \lambda$, where $\lambda \in T_1$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)|^2 - \frac{2n(k-1)}{k+1} \cdot \frac{|r(z)|^2|B'(z)|}{M_1^2 + M_2^2}\right\}^{1/2}(M_1^2 + M_2^2)^{1/2}, \tag{8}$$

where

$$M_1 = \max_{1 \leq i \leq n} |r(t_i)| \quad \text{and} \quad M_2 = \max_{1 \leq i \leq n} |r(s_i)|.$$

For $k = 1$, this reduces to Theorem D and generalizes a polynomial inequality due to Aziz [1, Theorem 4].

2. Lemmas.

For the proof of these theorems, we need the following lemmas. The first lemma is due to Li, Mohapatra and Rodrigues [6].

Lemma 1. *Suppose that $\lambda \in T_1$, then the following holds. The equation $B(z) - \lambda$ has exactly n simple roots (say) $t_1, t_2, \ldots, t_n$ and all lie on the unit circle T_1, and if $r \in R_n$ and $z \in T_1$, then*

$$B'(z)r(z) - r'(z)\left[B(z) - \lambda\right] = \frac{B(z)}{2}\sum_{k=1}^{n} C_k r(t_k)\left|\frac{B(z) - \lambda}{z - t_k}\right|^2, \tag{9}$$

where $C_k = C_k(\lambda)$ is defined by

$$C_k^{-1} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2} \qquad \text{for } k = 1, 2, \ldots, n. \tag{10}$$

Moreover for $z \in T_1$, we have

$$z\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} C_k \left|\frac{B(z) - \lambda}{z - t_k}\right|^2 \tag{11}$$

and also

$$|B'(z)| = z\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{|a_k|^2 - 1}{|z - a_k|^2}. \tag{12}$$

The next lemma which we need is due to autors [2].

Lemma 2. *Suppose* $t_1, t_2, \ldots, t_n$ *are the zeros of* $B(z) - \lambda$ *and* $s_1, s_2, \ldots, s_n$ *are the zeros of* $B(z) + \lambda$, *where* $\lambda \in T_1$. *If* $r \in R_n$ *and* $z \in T_1$, *then*

$$|r'(z)|^2 + |\left(r^*(z)\right)'|^2 < \frac{1}{2}|B'(z)|^2 \left\{ \left(\max_{1\le i\le n} |r(t_i)|\right)^2 + \left(\max_{1\le i\le n} |r(s_i)|\right)^2 \right\}. \tag{13}$$

We also need the following lemma which is due to Aziz and Zargar [3].

Lemma 3. *If* $z \in T_1$, *then*

$$\Re\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B(z)|}{2}. \tag{14}$$

3. Proofs of the theorems.

Proof. of Theorem 1. Let $r(z) = \dfrac{P(z)}{w(z)} \in R_n$. If $b_1, b_2, \ldots, b_m$ are the zeros of $P(z)$, then $m \le n$, $|b_j| \le k \le 1$, $j = 1, 2, \ldots, m$ and we have

$$\frac{zr'(z)}{r(z)} = \frac{zP'(z)}{P(z)} - \frac{zw'(z)}{w(z)} = \sum_{j=1}^{m} \frac{z}{z - b_j} - \frac{zw'(z)}{w(z)}. \tag{15}$$

Equation (15) with the help of Lemma 3 gives for $z \in T_1$,

$$\Re\left(\frac{zr'(z)}{r(z)}\right) = \Re\sum_{j=1}^{m} \frac{z}{z - b_j} - \Re\left(\frac{zw'(z)}{w(z)}\right)$$

$$= \sum_{j=1}^{m} \Re\left(\frac{z}{z - b_j}\right) - \left(\frac{n - |B'(z)|}{2}\right). \tag{16}$$

It can be easily verified that for $z \in T_1$, $|b| \le k \le 1$,

$$\Re\left(\frac{z}{z - b_k}\right) \ge \frac{1}{1 + k}. \tag{17}$$

Using inequality (17) in (16), we get for $z \in T_1$,

$$\Re\left(\frac{zr'(z)}{r(z)}\right) \ge \frac{m}{1 + k} - \frac{n - |B'(z)|}{2} = \frac{|B'(z)|}{2} + \frac{2m - n(1 + k)}{2(1 + k)}.$$

From which we obtain

$$\left|\frac{r'(z)}{r(z)}\right| \geq \Re\left(\frac{zr'(z)}{r(z)}\right) \geq \frac{|B'(z)|}{2} + \frac{2m - n(1+k)}{2(1+k)},$$

which is equivalent to inequality (6) and Theorem 1 is completely proved. ■

Proof. of Theorem 2. Let $r(z) = \dfrac{P(z)}{w(z)} \in R_n$. If $b_1, b_2, \ldots, b_m$ are the zeros of $P(z)$, then $m \leq n$, $|b_j| \geq k \geq 1$, $j = 1, 2, \ldots, m$ and we have

$$\frac{zr'(z)}{r(z)} = \frac{zP'(z)}{P(z)} - \frac{zw'(z)}{w(z)} = \sum_{j=1}^{m} \frac{z}{z - b_j} - \frac{zw'(z)}{w(z)}. \tag{18}$$

Equation (18) with the help of Lemma 3 gives for $z \in T_1$,

$$\Re\left(\frac{zr'(z)}{r(z)}\right) = \Re\sum_{j=1}^{m} \frac{z}{z - b_j} - \Re\frac{zw'(z)}{w(z)}$$

$$= \sum_{j=1}^{m} \Re\left(\frac{z}{z - b_j}\right) - \left(\frac{n - |B'(z)|}{2}\right). \tag{19}$$

It can be easily verified that for $z \in T_1$, $|b| \geq k \geq 1$,

$$\Re\left(\frac{z}{z - b_k}\right) \leq \frac{1}{1+k}. \tag{20}$$

Using inequality (20) in (19), we get for $z \in T_1$,

$$\Re\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{m}{1+k} - \frac{n - |B'(z)|}{2}$$

$$\leq \frac{n}{1+k} - \frac{n}{2} + \frac{|B'(z)|}{2} = \frac{|B'(z)|}{2} - \frac{n(k-1)}{2(k+1)}. \tag{21}$$

Also, if

$$r^*(z) = B(z)\overline{r(1/\bar{z})},$$

then

$$(r^*(z))' = B'(z)\overline{r(1/\bar{z})} - B(z)\left(\overline{r(1/\bar{z})}\right) \cdot \frac{1}{z^2}.$$

Since $z \in T_1$, we have $\bar{z} = 1/z$ and therefore,

$$\left|(r^*(z))'\right| = \left|zB'(z)\overline{r(z)} - B(z)\overline{zr'(z)}\right|$$

$$= \left| z\frac{B'(z)}{B(z)}\overline{r(z)} - \overline{zr'(z)} \right|. \tag{22}$$

Making use of (12) in equation (22), we obtain for $z \in T_1$,

$$\left|(r^*(z))'\right| = ||B'(z)|r(z) - zr'(z)|.$$

From which it follows that for $z \in T_1$,

$$\left|\frac{z\left(r^*(z)\right)'}{r(z)}\right|^2 = \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2$$

$$= |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|\Re\left(\frac{zr'(z)}{r(z)}\right). \tag{23}$$

Using inequality (21) in (23), we get

$$\left|\frac{z\left(r^*(z)\right)'}{r(z)}\right|^2 \geq |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - |B'(z)|\left\{|B'(z)| - \frac{n(k-1)}{k+1}\right\}$$

$$= \left|\frac{zr'(z)}{r(z)}\right|^2 + \frac{n(k-1)}{k+1}|B'(z)|.$$

Which implies for $z \in T_1$

$$|r'(z)|^2 + \frac{n(k-1)}{k+1}|r(z)|^2|B'(z)| \leq \left|(r^*(z))'\right|. \tag{24}$$

Inequality (24) in conjunction with Lemma 2, gives

$$2|r'(z)|^2 + \frac{n(k-1)}{k+1}|r(z)|^2|B'(z)| \leq \left|(r^*(z))'\right|^2 + |r'(z)|^2$$

$$\leq \frac{1}{2}|B'(z)|^2\left\{M_1^2 + M_2^2\right\}.$$

Equivalently

$$4|r'(z)|^2 \leq |B'(z)|^2(M_1^2 + M_2^2) - \frac{2n(k-1)}{k+1}|r(z)|^2|B'(z)|$$

$$= \left\{|B'(z)|^2 - \frac{2n(k-1)}{k+1}\frac{|r(z)|^2|B'(z)|}{M_1^2 + M_2^2}\right\}\left(M_1^2 + M_2^2\right)$$

which immediately leads to inequality (8) and this proves Theorem 2 completely.

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