Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 1-2

Variational Aspects of the Mixed Formulation for Fourth-Order Elliptic Eigenvalue Problems[1]

Andrey B. Andreev, Milena R. Racheva

Presented by P. Kenderov

A class of fourth-order elliptic equations with a spectral parameter in the domain as well as on the boundary are considered. This is a general kind of the eigenvalue problem for elliptic operator. We study such problems related to a saddle-point formulation. The corresponded variational eigenvalue problems in weak mixed formulation are discussed. The conditions when the variational bilinear forms are symmetric are presented. Finally, the theoretical results are illustrated by some examples.

AMS Subj. Classification: 65N25, 65N15.

Key words: fourth-order eigenvalue problems, mixed method

1. Introduction

Let Ω be a bounded domain in $R^d,\ d=1,2,3$ with a Lipschitz boundary $\Gamma=\partial\Omega$. Consider second-order strongly elliptic operators [4]

$$\mathcal{A}_i u(x) = p_i(x)\Delta u(x) + \sum_{j=1}^{d} q_{ij}(x)\partial_j u(x) + r_i(x)u(x), \quad i=1,2,$$

where Δ is the Laplace operator and $\partial_i u = \frac{\partial u}{\partial x_i},\ i=1,\dots,d,\ x=(x_1,\dots,x_d)$. Here and throughout the following, all the coefficients are the real-valued and continuously prolongable functions on $\overline{\Omega}$. We assume that $p_i(x)\in C^2(\Omega), q_{ij}(x)\in C^1(\Omega), r_i(x)\in C(\Omega), p_i(x)>0, q_{ij}(x)\geq 0, r_i(x)\geq 0, i=1,2; j=1,\dots,d$.

We consider the following fourth-order eigenvalue problem in mixed formulation:

$$\begin{aligned} \mathcal{A}_1 u(x) - \lambda c_1(x) u(x) &= \sigma(x), \\ \mathcal{A}_2 \sigma(x) - \lambda c_2(x) \sigma(x) &= b(x) u(x) + \lambda c_3(x) u(x), \quad x \in \Omega, \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{aligned} \alpha_1(s)\tfrac{\partial \sigma}{\partial n} + \beta_1(s)\tfrac{\partial u}{\partial n} + \gamma_1(s)\sigma + \delta_1(s) u &= \lambda(\varphi_1(s)\sigma + \psi_1(s) u), \\ \alpha_2(s)\tfrac{\partial \sigma}{\partial n} + \beta_2(s)\tfrac{\partial u}{\partial n} + \gamma_2(s)\sigma + \delta_2(s) u &= \lambda(\varphi_2(s)\sigma + \psi_2(s) u), \ s \in \Gamma, \end{aligned} \tag{2}$$

where the coefficients of (2) belong to $C(\Gamma)$ and $\frac{\partial}{\partial n}$ is the conormal derivative [3]. Without loss of generality if $\alpha_1(s) \neq 0$ then $\alpha_2(s) = 0$ because of the linear independence of boundary conditions (2) and $(\varphi_i, \psi_i) \neq (0,0)$, only if $(\alpha_i, \beta_i) \neq (0,0)$, $i = 1; 2$.

In the problem (1), (2) we look for number λ such that the solution (σ, u) of (1) is nontrivial and satisfies the boundary conditions (2). To this end, we can state this problem in mixed variational terms so that this statement will be completely equivalent to that in differential terms. More precisely, we assign two functionals (equations) such tha all stationary points of these functionals (and only these points) are eigenfunctions (σ, u) of the original problem (1), (2). In the mixed formulations (see [5, 10]), one has two unknown fields to be approximated by Galerkin procedure. An advantage of the mixed presentation is that one avoids the requirement of C^1-condition when the finite element method is used.

The fourth-order elliptic problem (1), (2) is of general kind when the eigenvalue parameter λ appears linearly in the boundary conditions. This type of problems arises in many physical situations. For example the eigenvalue parameter is mainly on the boundary when parabolic and hyperbolic equations with dynamical boundary conditions are considered [7, 9]. Typically, most homogeneous problems arising in the generalized method of eigenoscillations contain the spectral parameter in the boundary conditions [1].

In order to apply the various Galerkin type numerical methods, we have to present (1), (2) in the weak variational form. In this paper, we discuss some variational aspects of the fourth-order mixed formulation. The corresponding consideration for second-order equations and conventional formulations has already been extensively studied in the literature (see [1, 2, 9]).

2. Main Results

Our principal aim is to obtain some sufficient conditions concerning the mixed variational weak formulation of the problem (1), (2). In particular, for $d = 1$ and the conventional case, the necessary and sufficient conditions for selfadjointness of the fourth-order problems containing eigenvalue parameter in the boundary conditions are presented by the authors in [12].

Let $H^m(\Omega)$ be the usual Sobolev space for positive integer m (see for example [5]). Let $W \times V$ be a closed subspace of $H^1(\Omega) \times H^1(\Omega)$, i.e. $H_0^1(\Omega) \times H_0^1(\Omega) \subset W \times V \subseteq H^1(\Omega) \times H^1(\Omega)$ and $H_0^1(\Omega)$ being the first order Sobolev space on Ω composed of functions vanishing on $\Gamma = \partial\Omega$. Denoting by $(\cdot, \cdot)$ the L_2-inner product, we multiply the first equation of (1) by $\psi(x) \in W$ and the second one by $v(x) \in V$.

For notational convenience we shall drop the arguments x and s in the most cases. After integrating on Ω we get for any $\psi \in W,\ v \in V$

$$\begin{array}{rcl} (\mathcal{A}_1 u, \psi) - \lambda(c_1 u, \psi) & = & (\sigma, \psi), \\ (\mathcal{A}_2 \sigma, v) - \lambda(c_2 \sigma, v) & = & (bu, v) + \lambda(c_3 u, v). \end{array}$$

Consequently

$$(\mathcal{A}_1 u, \psi) + (\mathcal{A}_2 \sigma, v) - (\sigma, \psi) - (bu, v)$$

$$= \lambda \left\{ (c_1 u, \psi) + (c_2 \sigma, v) + (c_3 u, v) \right\}.$$

Using the boundary conditions (2) we rewrite the system in the form [5]:

$$a\left((\sigma, u), (\psi, v)\right) = \lambda b\left((\sigma, u), (\psi, v)\right), \quad \forall (\psi, v) \in W \times V. \tag{3}$$

The specification of a and $b-$forms will be made further. Our aim is to obtain the condition for the symmetry of the problem (3).

Having in mind that $n = (n_1, \ldots, n_d)$ and using Green's formula [11] it follows (here and throughout the following $\sum_j = \sum_{j=1}^d$):

$$(\mathcal{A}_1 u, \psi) + (\mathcal{A}_2 \sigma, v) - (\sigma, \psi) - (bu, v)$$

$$= \int_\Omega [p_1 \Delta u \psi + \sum_j q_{1j} \partial_j u \psi + r_1 u \psi] \, dx$$

$$+ \int_\Omega [p_2 \Delta \sigma v + \sum_j q_{2j} \partial_j \sigma v + r_2 \sigma v] \, dx - \int_\Omega \sigma \psi \, dx - \int_\Omega uv \, dx$$

$$= \int_\Gamma [p_1 \frac{\partial u}{\partial n}\psi + p_2 \frac{\partial \sigma}{\partial n} v + \sum_j q_{2j} n_j \sigma v]\, ds$$

$$+ \int_\Omega [-p_1 \nabla u \cdot \nabla \psi + \sum_j q_{1j}\partial_j u \psi + r_1 u\psi]\, dx$$

$$+ \int_\Omega [-p_2 \nabla \sigma \cdot \nabla v - \sum_j q_{2j}\sigma \partial_j v + r_2 \sigma v]\, dx - \int_\Omega \sigma\psi\, dx - \int_\Omega uv\, dx,$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$.

We look for the conditions when the expressions in the above equality are symmetric with respect to the couples (σ, u) and (ψ, v). Obviously, first of all this symmetry is available if

(4) $$p_1 = p_2 = p; \quad q_{1j} = -q_{2j} = q_j, \; j = 1, \ldots, d; \quad r_1 = r_2 = r.$$

In order to transform the integral on the boundary

$$\int_\Gamma [p_1 \frac{\partial u}{\partial n}\psi + p_2 \frac{\partial \sigma}{\partial n} v + \sum_j q_{2j} n_j \sigma v]\, ds$$

$$= \int_\Gamma [M((\sigma, u), (\psi, v)) + \lambda N((\sigma, u), (\psi, v))]\, ds,$$

we use the conditions (2), where M and N will be determined afterwards.

To assure the symmetry of the $b-$form in (3) it requires

(5) $$c_1 = c_2 = c.$$

Let us use the following denotation

$$\Delta_{\alpha,\beta}(s) = \begin{vmatrix} \alpha_1(s) & \beta_1(s) \\ \alpha_2(s) & \beta_2(s) \end{vmatrix}, \quad s \in \Gamma,$$

and we adopt the notations $\Delta_{\alpha,\gamma}$, $\Delta_{\beta,\gamma}$, $\Delta_{\alpha,\varphi}$ and so on for other determinants, respectively.

Theorem 1. *Let the coefficients of (1), (2) satisfy the conditions of smoothness determined in the previous section. If (4), (5) and the equations*

(6) $$p(\Delta_{\alpha,\delta} + \Delta_{\beta,\gamma}) = \sum_j q_j n_j \Delta_{\alpha,\beta},$$

(7) $$\Delta_{\alpha,\psi} + \Delta_{\beta,\varphi} = 0$$

are fulfilled, the eigenvalue problem (3) in mixed formulation is symmetric.

Proof.

Case 1 First, let the eigenvalue parameter λ appear linearly in the boundary conditions. Then $\alpha_1(s) \neq 0$ and $\alpha_2(s) = 0$.

Case 1.1 $\Delta_{\alpha,\beta} \neq 0$, i.e. $\beta_2(s) \neq 0$.

By calculating the conormal derivatives (we drop the argument $s \in \Gamma$)

$$\begin{aligned} \tfrac{\partial\sigma}{\partial n} &= \tfrac{1}{\Delta_{\alpha,\beta}} \left\{\Delta_{\beta,\gamma}\sigma + \Delta_{\beta,\delta}u - \lambda\Delta_{\beta,\varphi}\sigma - \lambda\Delta_{\beta,\psi}u\right\}, \\ \tfrac{\partial u}{\partial n} &= \tfrac{1}{\Delta_{\alpha,\beta}} \left\{-\Delta_{\alpha,\gamma}\sigma - \Delta_{\alpha,\delta}u + \lambda\Delta_{\alpha,\varphi}\sigma + \lambda\Delta_{\alpha,\psi}u\right\}, \end{aligned}$$

we obtain

$$p\frac{\partial u}{\partial n}\psi + p\frac{\partial \sigma}{\partial n}v - \sum_j q_j n_j \sigma v$$

$$= \frac{p}{\Delta_{\alpha,\beta}}\left\{-\Delta_{\alpha,\gamma}\sigma\psi - \Delta_{\alpha,\delta}u\psi + \Delta_{\beta,\gamma}\sigma v + \Delta_{\beta,\delta}uv\right\} - \sum_j q_j n_j \sigma v$$

$$+\frac{\lambda p}{\Delta_{\alpha,\beta}}\left\{\Delta_{\alpha,\varphi}\sigma\psi + \Delta_{\alpha,\psi}u\psi - \Delta_{\beta,\varphi}\sigma v - \Delta_{\beta,\psi}uv\right\}.$$

One can untroduce the expressions for M and N:

$$M = \frac{1}{\Delta_{\alpha,\beta}}\left\{-p\Delta_{\alpha,\gamma}\sigma\psi - p\Delta_{\alpha,\delta}u\psi + (p\Delta_{\beta,\gamma} - \Delta_{\alpha,\beta}\sum_j q_j n_j)\sigma v + p\Delta_{\beta,\delta}uv\right\},$$

$$N = \frac{p}{\Delta_{\alpha,\beta}}\left\{\Delta_{\alpha,\varphi}\sigma\psi + \Delta_{\alpha,\psi}u\psi - \Delta_{\beta,\varphi}\sigma v - \Delta_{\beta,\psi}uv\right\}.$$

It is easy to see that the conditions (6) and (7) ensure symmetry of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in (3) respectively.

Case 1.2 $\Delta_{\alpha,\beta} = 0$ and $\Delta_{\alpha,\gamma} \neq 0$, i.e. $\beta_2(s) = 0,\ \gamma_2(s) \neq 0$. Then

$$\begin{aligned} \tfrac{\partial\sigma}{\partial n} &= \tfrac{1}{\Delta_{\alpha,\gamma}} \left\{-\Delta_{\beta,\gamma}\tfrac{\partial u}{\partial n} + \Delta_{\gamma,\delta}u + \lambda\tfrac{\Delta_{\alpha,\delta}}{\Delta_{\alpha,\gamma}}u - \lambda\Delta_{\gamma,\psi}u\right\}, \\ \sigma &= -\tfrac{\Delta_{\alpha,\delta}}{\Delta_{\alpha,\gamma}}u. \end{aligned}$$

Thus, we obtain

$$p\frac{\partial u}{\partial n}\psi + p\frac{\partial \sigma}{\partial n}v - \sum_j q_j n_j \sigma v$$

$$= \frac{p}{\Delta_{\alpha,\gamma}} \left\{ -\Delta_{\alpha,\delta} \frac{\partial u}{\partial n} v - \Delta_{\beta,\gamma} \frac{\partial u}{\partial n} v + \Delta_{\gamma,\delta} uv \right\} + \frac{1}{\Delta_{\alpha,\gamma}} \sum_j q_j n_j \Delta_{\alpha,\delta} uv$$

$$+ \frac{\lambda p}{\Delta_{\alpha,\gamma}} \left\{ \frac{\Delta_{\gamma,\varphi} \Delta_{\alpha,\delta}}{\Delta_{\alpha,\gamma}} - \Delta_{\gamma,\psi} \right\} uv.$$

In this case, it follows

$$M = \frac{1}{\Delta_{\alpha,\gamma}} \left\{ -p(\Delta_{\alpha,\delta} + \Delta_{\beta,\gamma}) \frac{\partial u}{\partial n} v + (p\Delta_{\gamma,\delta} + \sum_j q_j n_j \Delta_{\alpha,\delta}) uv \right\},$$

$$N = \frac{p}{\Delta_{\alpha,\gamma}} \left\{ \frac{\Delta_{\gamma,\varphi} \Delta_{\alpha,\delta}}{\Delta_{\alpha,\gamma}} - \Delta_{\gamma,\psi} \right\} uv.$$

The condition (6), with regard to the assumption $\Delta_{\alpha,\beta} = 0$ (and, consequently, $\Delta_{\alpha,\varphi} = \Delta_{\alpha,\psi} = 0$), gives

$$p\Delta_{\alpha,\delta} + p\Delta_{\beta,\gamma} = 0.$$

Case 1.3 $\Delta_{\alpha,\beta} = \Delta_{\alpha,\gamma} = 0,\ \Delta_{\alpha,\delta} \neq 0$.

Such being the case, the boundary conditions have got the following representation:

$$\begin{aligned} \alpha_1 \tfrac{\partial \sigma}{\partial n} + \beta_1 \tfrac{\partial u}{\partial n} + \gamma_1 \sigma &= \lambda \varphi_1 \sigma, \\ u &= 0. \end{aligned}$$

Consequently

$$p\frac{\partial u}{\partial n} \psi + p\frac{\partial \sigma}{\partial n} v - \sum_j q_j n_j \sigma v = p\frac{\partial u}{\partial n} \psi.$$

Then the condition (6) is not satisfied and the problem is not symmetric.

Case 2 Let us consider the cases when $\alpha_1(s) = \alpha_2(s) = 0,\ \beta_1(s) \neq 0,\ \beta_2(s) = 0$ and $\varphi_2(s) = \psi_2(s) = 0$.

Case 2.1 $\Delta_{\beta,\gamma} \neq 0$, i.e. $\gamma_2(s) \neq 0$.

We have

$$\begin{aligned} \tfrac{\partial u}{\partial n} &= \tfrac{1}{\Delta_{\beta,\gamma}} \left\{ \Delta_{\gamma,\delta} u - \lambda \tfrac{\Delta_{\gamma,\varphi} \Delta_{\beta,\delta}}{\Delta_{\beta,\gamma}} u + \lambda \Delta_{\gamma,\psi} u \right\}, \\ \sigma &= -\tfrac{\Delta_{\beta,\delta}}{\Delta_{\beta,\gamma}} u. \end{aligned}$$

We obtain

$$p\frac{\partial u}{\partial n}\psi + p\frac{\partial \sigma}{\partial n}v - \sum_j q_j n_j \sigma v$$

$$= \frac{p}{\Delta_{\beta,\gamma}}\left\{-\frac{\Delta_{\gamma,\delta}\Delta_{\beta,\delta}}{\Delta_{\beta,\gamma}}uv + \Delta_{\beta,\gamma}\frac{\partial \sigma}{\partial n}v\right\} + \frac{\Delta_{\beta,\delta}}{\Delta_{\beta,\gamma}}\sum_j q_j n_j uv$$

$$-\frac{\lambda p}{\Delta_{\beta,\gamma}}\left\{\frac{\Delta_{\gamma,\varphi}\Delta_{\beta,\delta}}{\Delta_{\beta,\gamma}} - \Delta_{\gamma,\psi}\right\}uv.$$

Let us denote

$$M = \frac{\Delta_{\beta,\delta}}{\Delta_{\beta,\gamma}}\left\{\sum_j q_j n_j - p\frac{\Delta_{\gamma,\delta}}{\Delta_{\beta,\gamma}}\right\}uv + p\frac{\partial \sigma}{\partial n}v,$$

$$N = -\frac{p}{\Delta_{\beta,\gamma}}\left\{\frac{\Delta_{\gamma,\varphi}\Delta_{\beta,\delta}}{\Delta_{\beta,\gamma}} - \Delta_{\gamma,\psi}\right\}uv.$$

We see that in this case the condition (6) is not fulfilled.

Case 2.2 $\Delta_{\beta,\gamma} = 0,\ \Delta_{\beta,\delta} \neq 0$, i.e. $\delta_2(s) \neq 0$. Then the boundary conditions are:

$$\beta_1 \tfrac{\partial u}{\partial n} + \gamma_1\sigma = \lambda\varphi_1\sigma,$$

$$u = 0,$$

it follows

$$p\frac{\partial u}{\partial n}\psi + p\frac{\partial \sigma}{\partial n}v - \sum_j q_j n_j \sigma v$$

$$= \frac{p}{\beta_1}\left\{-\gamma_1 + \lambda\varphi_1\right\}\sigma\psi,$$

and the conditions (6), (7) are satisfied.

Case 3 Let $(\alpha_1(s),\alpha_2(s)) = (\beta_1(s),\beta_2(s)) = (0,0),\ \gamma_1(s) \neq 0$. Then $\gamma_2(s) = 0$ and $(\varphi_i(s),\psi_i(s) = (0,0),\ i = 1,2$.

Actually the boundary conditions, corresponding to the coefficients are $u = 0,\ \sigma = 0$ and

$$p\frac{\partial u}{\partial n}\psi + p\frac{\partial \sigma}{\partial n}v - \sum_j q_j n_j \sigma v = 0.$$

■

R e m a r k T he result of the theorem is valid if $\Gamma = \cup_k \Gamma_k$ with $meas\,(\Gamma_k) \neq 0$, subject to the conditions (6), (7) are fulfilled on each part Γ_k of the boundary.

This theorem gives sufficient conditions in order to present the fourth-order eigenvalue problem in a symmetric mixed formulation. Then the equation (1) has the following general presentation:

$$\begin{aligned} p\Delta u + \textstyle\sum_j q_j \partial_j u + ru - \lambda cu &= \sigma, \\ p\Delta \sigma - \textstyle\sum_j q_j \partial_j \sigma + r\sigma - \lambda c\sigma &= bu + \lambda c_3 u. \end{aligned}$$

Consequently, in this case the original differential equation is:

$$\begin{aligned} p^2\Delta^2 u + 2pr\Delta u - \sum_{i,j=1}^{d} q_i q_j \partial_{ij}^2 u + (r^2 - b)u \\ = \lambda\,[2pc\Delta u + 2rcu + c_3 u] - \lambda^2 c^2 u. \end{aligned}$$

This shows that our considerations include cases when eigenvalue parameter λ is in a nonlinear term (subject to $c(x) \neq 0$), i.e. the equation has a term containing λ^2. We would like to emphasize that the mixed method transforms this term to the linear case. The eigenvalue problems in which the eigenvalue appears nonlinearly are mostly the Sturm-Liouville equations [8].

It is clear that

$$a((\sigma,u),(\psi,v)) = \lambda b((\sigma,u),(\psi,v)), \quad \forall (\psi,v) \in W \times V,$$

where

$$\begin{aligned} a((\sigma,u),(\psi,v)) = \int_\Omega [-p(\nabla u\cdot\nabla\psi + \nabla\sigma\cdot\nabla v + \sum_j q_j(\partial_j u\ \psi + \sigma\partial_j\ v) \\ +r(u\psi + \sigma v)]\,dx - \int_\Omega (\sigma\psi + buv)\,dx + \int_\Gamma M((\sigma,u),(\psi,v))\,ds, \end{aligned}$$

$$b((\sigma,u),(\psi,v)) = \int_\Omega [c(u\psi + \sigma v) + c_3 uv]\,dx - \int_\Gamma N((\sigma,u),(\psi,v))\,ds.$$

3. Examples

In this section we shall present two examples illustrating the applicability of the proved theorem.

Example 1

The first example deals with a bending thin plate on elastic foundation [6].

$$\Delta^2 u + Cu = \lambda u, \tag{8}$$

with a clamped boundary

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{9}$$

The natural frequencies of the plate are determined by the eigenvalues λ of the differential system (8), (9).

In order to use the theorem we determine the parameters: $p = -1$; $r = 0$; $q_j = 0$; $b = C$; $c_3 = 1$ and $c = 0$. It is easy to see that the conditions (6), (7) are fulfilled. The mixed formulation is:

$$\begin{aligned} -\Delta u &= \sigma \\ -\Delta \sigma &= Cu + \lambda u. \end{aligned}$$

For the symmetric variational form we obtain:

$$\int_\Omega [\nabla u \cdot \nabla \psi + \nabla \sigma \cdot \nabla v - \sigma \psi - Cuv]\, dx$$

$$= \lambda \int_\Omega uv\, dx \quad \forall (\psi, v) \in W \times V.$$

R e m a r k T he problem (8), (9) corresponds to the case 2.2 of the theorem. If the thin plate is simply supported, i.e. $u = \Delta u = 0$ on the boundary, that is the case 3 of the theorem and the similar conclusions are available.

Example 2

Let us consider bending oscillations of a beam with fixed ends [6]:

$$(\alpha y'')'' = \lambda \rho F y, \quad x \in (0, l),$$

$$y = y' = 0, \quad x = 0; l,$$

where $\alpha > 0$ is the flexural rigidity, ρ is the density and F is the area of transversal section of the beam. Consequently, ρF is the mass per unit length.

We determine the coefficients corresponding to the theorem:

$$p = \sqrt{\alpha};\ r = \frac{q^2}{2p} = \frac{q^2}{2\sqrt{\alpha}};\ b = r^2 = \frac{q^4}{4\alpha};\ c_3 = \rho F;\ c = 0.$$

This is again the case 2.2 of the theorem. It should be noted that one can introduce a real parameter $q \in \mathbf{R}$ which has a special part in the mixed presentation

$$\begin{array}{rcl} -\sqrt{\alpha}u'' + qu' - \frac{q^2}{2\sqrt{\alpha}}u & = & \sigma \\ -\sqrt{\alpha}\sigma'' - q\sigma' - \frac{q^2}{2\sqrt{\alpha}}\sigma & = & \frac{q^4}{4\alpha}u + \lambda\rho F u \end{array}$$

In other words, for any $q \in \mathbf{R}$ we obtain a symmetric variational formulation. For this, the parameter q may be used to construct computational schemes of mixed variational type.

References

[1] Agranovich M. S., B. Z. Katsenelenbaum, A. N. Sivov and N. N. Voitovich, *Generalized Method for Eigenoscillations in Diffraction Theory* - WILEY-VCH Verlag Berlin GmbH, Berlin, 1999.

[2] A. B. Andreev, A. H. Hristov, On the variational aspects for elliptic problems with parameter on the boundary, *Advances in Numer. Methods and Applications II*, World Scientific, 587-593, 1998.

[3] I. Babuška, A. K. Aziz, Survey Lectures on the Mathematical Foundations of the Finite Element Method, it The Math. Foundations of the FEM with Applications to PDE, A.K.Aziz ed., Academic Press, New York, 1972.

[4] I. Babuška, J. Osborn, Eigenvalue Problems, *Handbook of Numer. Anal., Vol.II*, North-Holland, Amsterdam, 1991.

[5] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, Oxford, 1978.

[6] L. Collatz, *Eigenwertaufgaben mit Technischen Anwendungen*, Leipzig, Acad. Verlag, 1963.

[7] J. Escher, Quasilinear parabolic systems with dynamical boundary conditions, *Commun. in PDEs*, **18(788)** (1993), 1309-1364.

[8] L. Greenberg and I. Babuška, A continuous analogue of Sturm sequences in the context of Sturm-Liouville equations, *SIAM J. Numer. Anal.*, **26** (1989), No 4, 920-945.

[9] D. B. Hinton, J. K. Shaw, Differential operators with spectral parameter incompletely in the boundary conditions, *Funkcialaj Ekvacioj (Serio Internacia)*, **33** (1990), 363-385.

[10] B. Mercier, Numerical solution of the biharmonic problem by mixed finite elements of class C^0, *Bull. Un. Math. Ital.*, bf 10 (1974), 133-149.

[11] S. G. Mikhlin, *Variational Methods in Mathematical Physics*, Oxford: Pergamon, 1964.

[12] M. R. Racheva, A. B. Andreev, Variational aspects of one-dimensional fourth-order problems with eigenvalue parameter in the boundary conditions, *Sib. JNM*, **5** (2002), No4(5), 373-380.

Received: 03.04.2003

e-mail: Andreev@tugab.bg
MRacheva@mail.com