Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 1-2

Modified Szasz-Mirakyan Operators

L. Rempulska, Z. Walczak

Presented by P. Kenderov

In this paper we introduce modified Szasz-Mirakyan operators and we give three approximation theorems for them.

This paper was motivated by results given in [1] and [3].

A.M.S. Subject classification 41A36

Key words: Szasz-Mirakyan operator, approximation theorem, Voronovskaya theorem.

1. Introduction

1.1. Let C_B be the set of all real-valued functions f uniformly continuous and bounded on $R_0 := [0, +\infty]$ and let the norm be defined by

$$\|f\| \equiv \|f(\cdot)\| := \sup_{x \in R_0} |f(x)|. \tag{1}$$

For a fixed $r \in N_0 := \{0, 1, 2, \dots\}$ we denote by C_B^r the set of all $f \in C_B$ with derivatives $f', \dots, f^{(r)}$ belonging also to C_B ($C_B^0 \equiv C_B$). The norm in C_B^r is given by (1).

In [1] were examined approximation properties of Szasz-Mirakyan operators

$$S_n(f; x) := \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \qquad x \in R_0, \quad n \in N := \{1, 2, \cdots\}, \tag{2}$$

where

$$p_k(t) := e^{-t} \frac{t^k}{k!}, \qquad t \in R_0, \quad k \in N_0. \tag{3}$$

The direct theorem given in [1] yield the following inequality for $f \in C_B$:

$$(4) \qquad |S_n(f;x) - f(x)| \le M_1\omega_2\left(f;\sqrt{x/n}\right), \qquad x \in R_0, \quad n \in N,$$

where $\omega_2(f;\cdot)$ is the second modulus of smoothness of f and $M_1 = const. > 0$. The inequality (4) implies that

$$(5) \qquad \lim_{n\to\infty} S_n(f;x) = f(x),$$

for every $x \in R_0$ and $f \in C_B$. Moreover it is known ([1], [2]) that if $f \in C_B^2$, then

$$(6) \qquad \lim_{n\to\infty} n(S_n(f;x) - f(x)) = \frac{x}{2}f''(x), \qquad \text{for } x \in R_0.$$

1.2. In this paper we shall prove that the approximation order given in (4) can be improved for $f \in C_B^r$ by certain modification of $S_n(f)$.

Definition. *Let $r \in N_0$ be a fixed number. For $f \in C_B^r$ and $n \in N$ we define operators:*

$$S_{n,r}(f;x) := \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!}\left(x - \frac{k}{n}\right)^j, \qquad x \in R_0,$$

where $p_k(\cdot)$ is defined by (3).

Clearly $S_{n,0}(f;x) = S_n(f;x)$ for $x \in R_0$, $n \in N$ and $f \in C_B$.

In Section 2 we shall give some elementary properties of $S_{n,r}(f)$. The main theorems we shall give in Section 3.

In this paper we shall denote by $M_k(a)$, $k = 1, 2, \ldots$, the suitable positive constants depending only on a.

2. Lemmas

It is known ([1]) that

$$(8) \qquad S_n(1;x) = 1, \qquad S_n(t-x;x) = 0,$$

$$S_n\left((t-x)^{q+1};x\right) = \frac{x}{n}\left\{S_n'\left((t-x)^q;x\right) + qS_n\left((t-x)^{q-1};x\right)\right\},$$

for $x \in R_0$, $n \in N$ and $q \in N$.

Using mathematical induction for $q \in N$ and by (8) and (9), we can prove the following

Lemma 1. *For every $2 \leq q \in N$ we have*

$$S_n\left((t-x)^q;x\right) = \sum_{j=1}^{[q/2]} b_{j,q}\frac{x^j}{n^{q-j}}, \qquad x \in R_0, \quad n \in N,$$

where $b_{j,q}$ are positive numerical coefficients depending only on j and q ($[y]$ denotes the integral part of $y \in R_0$).

Applying Lemma 1, we immediately obtain

Lemma 2. *For every $q \in N$ there exists a positive constant $M_1(q)$ such that*

$$\sup_{x\in R_0} (1+x^q)^{-1} S_n\left((t-x)^{2q};x\right) \leq M_1(q)n^{-q}, \qquad n \in N.$$

In general

$$\sup_{x\in R_0} \left(1+x^{q/2}\right)^{-1} \left|S_n\left((t-x)^q;x\right)\right| \leq M_2(q)n^{-[(q+1)/2]}, \qquad n \in N.$$

Applying Lemma 2, we shall prove the main lemma.

Lemma 3. *Let $r \in N_0$ be fixed number. Then there exists a positive constant $M_3(r)$ such that*

$$\sup_{x\in R_0} \left(1+x^{r/2}\right)^{-1} \left|S_{n,r}\left(f;x\right)\right| \leq M_3(r)\sum_{j=0}^{r}\frac{\|f^{(j)}\|}{j!}, \tag{10}$$

for all $f \in C_B^r$ and $n \in N$.

The formula (7) and (10) show that $S_{n,r}(f)$ is well-defined on the space C_B^r and the function $\left(1+x^{r/2}\right)^{-1} S_{n,r}(f;x)$ belongs to C_B.

Proof. If $r = 0$, then by (7) and (1) we have

$$|S_{n,0}(f;x)| \leq \|f\| \sum_{k=0}^{\infty} p_k nx = \|f\|, \qquad \text{for } x \in R_0, n \in N.$$

If $r \geq 1$, then by (2), (3), (7) and Lemma 2 we can write

$$S_{n,r}(f;x) = \sum_{j=0}^{r} \frac{1}{j!} S_n\left(f^{(j)}(t)(x-t)^j;x\right).$$

Next, by Hölder inequality and (1) and (8) we get

$$|S_n\left(f^{(j)}(t)(x-t)^j;x\right)| \leq S_n\left(|f^{(j)}(t)(x-t)^j|;x\right) \leq$$

$$\leq \|f^{(j)}\| S_n\left(|t-x|^j;x\right) \leq \|f^{(j)}\| \left(S_n\left((t-x)^{2j};x\right)\right)^{1/2}.$$

From the above and by Lemma 2 we obtain

$$\left(1+x^{r/2}\right)^{-1} |S_{n,r}(f;x)| \leq \sum_{j=0}^{r} \frac{\|f^{(j)}\|}{j!} \left\{(1+x^r)^{-1} S_n\left((t-x)^{2j};x\right)\right\}^{1/2} \leq$$

$$\leq M_4(r) \sum_{j=0}^{r} \frac{\|f^{(j)}\|}{j!} n^{-j/2},$$

for all $x \in R_0$ and $n \in N$. Thus the proof of (10) is completed.

We remark that if $f(x) = x^q$, $x \in R_0$, $q \in N_0$, then by the Taylor formula it follows that

$$f(x) = \sum_{j=0}^{q} \frac{f^{(j)}(y)}{j!}(x-y)^j, \qquad x \in R_0,$$

for every fixed $y \in R_0$. This fact and (7) yield

Lemma 4. *Let $f(x) = x^q$, $x \in R_0$, $q \in N_0$. Then for every fixed $q \leq\in N_0$ we have*

$$S_{n,r}(t^q;x) = x^q, \qquad x \in R_0,\ n \in N.$$

3. Theorems

3.1. First we shall prove an analogy of estimation (4). Let $\omega_1(f;\cdot)$ be the modulus of continuity of $f \in C_B$, i.e.

(11) $$\omega_1(f;t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|, \qquad t \geq 0,$$

where $\Delta_h f(x) := f(x+h) - f(x)$ for $x, h \in R_0$.

Theorem 1. *Let $r \in N_0$ be a fixed number. Then there exists a positive constant $M_5(r)$ such that*

$$\sup_{x\in R_0}\left(1+x^{(r+1)/2}\right)^{-1}\left|S_{n,r}(f;x)-f(x)\right| \le M_5(r)n^{-r/2}\omega_1\left(f^{(r)};n^{-1/2}\right) \tag{12}$$

for every $f \in C_B^r$ and $n \in N$.

P r o o f. The inequality (12) for $r = 0$ follows from (4).

Let $f \in C_B^r$ with $r \ge 1$ and let $y \in R_0$ be a fixed point. We apply the following modified Taylor formula

$$f(x) = \sum_{j=0}^{r}\frac{f^{(j)}(y)}{j!}(x-y)^j +$$

$$+\frac{(x-y)^r}{(r-1)!}\int_0^1(1-t)^{r-1}\left\{f^{(r)}(y+t(x-y))-f^{(r)}(y)\right\}dt, \qquad x \in R_0.$$

Setting $y = \frac{k}{n}$ for fixed $k \in N_0$ and $n \in N$, we derive the following equality from (7) and (8):

$$f(x) = \sum_{k=0}^{\infty}\left(p_k(nx)f(x)\right) = S_{n,r}(f(t);x) \tag{13}$$

$$+\sum_{k=0}^{\infty}p_k(nx)\frac{\left(x-\frac{k}{n}\right)^r}{(r-1)!}\int_0^1(1-t)^{r-1}\left\{f^{(r)}\left(\frac{k}{n}+t\left(x-\frac{k}{n}\right)\right)-f^{(r)}\left(\frac{k}{n}\right)\right\}dt,$$

for $x \in R_0$ and $n \in N$. Applying (11) and the inequality $\omega_1(g;\lambda t) \le (1+\lambda)\omega_1(g;t)$ for $g \in C_B$ and $\lambda, t \ge 0$, we get

$$\left|f^{(r)}\left(\frac{k}{n}+t\left(x-\frac{k}{n}\right)\right)-f^{(r)}\left(\frac{k}{n}\right)\right| \le \omega_1\left(f^{(r)};t\left|x-\frac{k}{n}\right|\right) \le$$

$$\le \omega_1\left(f^{(r)};\left|x-\frac{k}{n}\right|\right) \le \omega_1\left(f^{(r)};n^{-1/2}\right)\left(1+n^{1/2}\left|x-\frac{k}{n}\right|\right)$$

for $0 \le t \le 1$ and $x \in R_0$, $k \in N_0$, $n \in N$. This inequality and (13) and (2) imply that

(14)

$$\left|f(x)-S_{n,r}(f(t);x)\right| \le \omega_1\left(f^{(r)};n^{-1/2}\right)\sum_{k=0}^{\infty}p_k(nx)\frac{\left|x-\frac{k}{n}\right|^r}{r!}\left(1+n^{1/2}\left|x-\frac{k}{n}\right|\right) =$$

$$= \omega_1\left(f^{(r)}; n^{-1/2}\right) \frac{1}{r!}\left\{S_n\left(|t-x|^r; x\right) + n^{1/2} S_n\left(|t-x|^{r+1}; x\right)\right\}$$

for all $x \in R_0$ and $n \in N$. Further by Hölder inequality and Lemma 2 and (8) we have

(15)

$$S_n\left(|t-x|^q\right) \le \left(S_n\left((t-x)^{2q}\right)\right)^{1/2} \le \left(M_1(q)\frac{x^q}{n^q}\right)^{1/2}, \qquad x \in R_0,\ n, q \in N.$$

Using (15) to (14), we obtain

$$\left(1 + x^{(r+1)/2}\right)^{-1} |S_{n,r}(f; x) - f(x)| \le M_6(r) n^{-r/2} \omega_1\left(f^{(r)}; n^{-1/2}\right),$$

for $x \in R_0$ and $n \in N$. This completes the proof of (12).

From Theorem 1 we derive the following two corollaries.

Corollary 1. *If $f \in C^r_B$, $r \in N_0$, then*

$$\lim_{n\to\infty} n^{r/2}\{S_{n,r}(f; x) - f(x)\} = 0 \qquad \text{for } x \in R_0.$$

Corollary 2. *If $f \in C^r_B$, $r \in N_0$, and $f^{(r)} \in Lip\alpha$ with $0 < \alpha \le 1$, i.e. $\omega_1\left(f^{(r)}; t\right) = O(t^\alpha)$ for $t > 0$, then*

$$\sup_{x\in R_0} \left(1 + x^{(r+1)/2}\right)^{-1} |S_{n,r}(f; x) - f(x)| = O(n^{-(r+\alpha)/2}), \qquad n \in N.$$

3.2. Now we shall give the Voronovskaya type theorem.

Theorem 2. *Suppose that $f \in C^{r+2}_B$ with a fixed $r \in N_0$. Then for every $x \in R_0$ we have*

(16) $$S_{n,r}(f; x) - f(x) = \frac{(-1)^r f^{(r+1)}(x) S_n\left((t-x)^{r+1}; x\right)}{(r+1)!} +$$

$$+ \frac{(-1)^r (r+1) f^{(r+2)}(x) S_n\left((t-x)^{r+2}; x\right)}{(r+2)!} + o_x\left(\frac{1}{n^{1+r/2}}\right),$$

as $n \to \infty$.

Proof. From (7) we get $S_{n,r}(f; 0) = f(0)$, $n \in N$, $r \in N_0$.

Fix $x > 0$. If $f \in C_B^{r+2}$, then $f^{(j)} \in C_B^{r+2-j}$, $0 \le j \le r$. Hence for every $f^{(j)}$ we can write the Taylor formula:

$$f^{(j)}(t) = \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^i + \varphi_j(t;x)(t-x)^{r+2-j}, \qquad 0 \le j \le r, \tag{17}$$

for $t \in R_0$, where $\varphi_j(t) \equiv \varphi_j(t;x)$ is function such that $\varphi_j(t)t^{r+2-j}$ belongs to C_B^{r+2-j} and $\lim_{t\to x} \varphi_j(t) = 0$. Setting $t = \frac{k}{n}$ in (17) and using to (7), we get

$$\begin{aligned} S_{n,r}(f;x) &= \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r} \frac{\left(x-\frac{k}{n}\right)^j}{j!} \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}\left(\frac{k}{n}-x\right)^i + \\ &+ \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r} \frac{\left(x-\frac{k}{n}\right)}{j!} \varphi_j\left(\frac{k}{n};x\right)\left(\frac{k}{n}-x\right)^{r+2-j} := \\ &:= A_{n,r}(x) + B_{n,r}(x), \quad n \in N. \end{aligned} \tag{18}$$

We observe that

$$\begin{aligned} A_{n,r}(x) &= \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r} \frac{\left(x-\frac{k}{n}\right)^j}{j!} \sum_{l=j}^{r+2} \frac{f^{(l)}(x)}{(l-j)!}\left(\frac{k}{n}-x\right)^{l-j} = \\ &= \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r} \frac{(-1)^j}{j!} \left\{ \sum_{l=j}^{r} \frac{f^{(l)}(x)}{(l-j)!}\left(\frac{k}{n}-x\right)^{l} + \right. \\ &\left. + \frac{f^{(r+1)}(x)}{(r+1-j)!}\left(\frac{k}{n}-x\right)^{r+1} + \frac{f^{(r+2)}(x)}{(r+2-j)!}\left(\frac{k}{n}-x\right)^{r+2} \right\} = \\ &= \sum_{k=0}^{\infty} p_k(nx) \sum_{l=0}^{r} \frac{f^{(l)}(x)}{l!}\left(\frac{k}{n}-x\right)^{l} \sum_{j=0}^{l} \binom{l}{j}(-1)^j + \\ &+ \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{k=0}^{\infty} p_k(nx)\left(\frac{k}{n}-x\right)^{r+1} \sum_{j=0}^{r} \binom{r+1}{j}(-1)^j + \\ &+ \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{k=0}^{\infty} p_k(nx)\left(\frac{k}{n}-x\right)^{r+2} \sum_{j=0}^{r} \binom{r+2}{j}(-1)^j \end{aligned}$$

for $n \in N$. It is easily verified that for every $r \in N$ we have

$$\sum_{j=0}^{r} \binom{r}{j}(-1)^j = 0, \qquad \sum_{j=0}^{r} \binom{r+1}{j}(-1)^j = (-1)^r, \tag{19}$$

$$\sum_{j=0}^{r} \binom{r+2}{j} (-1)^j = (r+1)(-1)^r.$$

From the above and by (2) and (3) we deduce that

$$A_{n,r}(x) = f(x) + \frac{(-1)^r f^{(r+1)}(x)\, S_n\left((t-x)^{r+1}; x\right)}{(r+1)!} + \tag{20}$$

$$+ \frac{(-1)^r (r+1) f^{(r+2)}(x)\, S_n\left((t-x)^{r+2}; x\right)}{(r+2)!}, \qquad n \in N.$$

Arguing as in the proof of Lemma 3, we get

$$B_{n,r}(x) = \sum_{k=0}^{\infty} p_k(nx) \left(\frac{k}{n} - x\right)^{r+2} \Phi_r\left(\frac{k}{n}; x\right) = S_n\left((t-x)^{r+2} \Phi_r(t); x\right),$$

for $n \in N$, where

$$\Phi_r(t) \equiv \Phi_r(t; x) := \sum_{j=0}^{r} \frac{(-1)^j}{j!} \varphi_j(t; x), \qquad t \in R_0,$$

and Φ_r is function belonging to C_B and $\lim_{t \to x} \Phi_r(t) = \Phi_r(x) = 0$. Applying Hölder inequality and by Lemma 2, we can write

$$|B_{n,r}(x)| \leq \left(S_n\left(\Phi_r^2(t); x\right)\right)^{1/2} \left(S_n\left((t-x)^{2r+4}; x\right)\right)^{1/2} \leq$$

$$\leq \left(M_1(q) \left(\frac{x}{n}\right)^{r+2}\right)^{1/2} \left(S_n\left(\Phi_r^2(t); x\right)\right)^{1/2}, \qquad n \in N.$$

Since $\Phi_r^2 \in C_B$, we have by (5)

$$\lim_{n \to \infty} S_n\left(\Phi_r^2(t); x\right) = \Phi_r^2(x) = 0.$$

From the above we deduce that

$$B_{n,r}(x) = o_x\left(\frac{1}{n^{1+r/2}}\right), \qquad \text{as } n \to \infty. \tag{21}$$

Collecting (18), (20) and (21) we obtain (16).

Theorem 2 and Lemma 1 imply the following

Corollary 3. *Let $f \in C_B^{r+2}$, $r \in N$. Then there exists a positive constant $M_7(r)$ such that*

$$\lim_{n\to\infty} n^{[1+r/2]}\{S_{n,r}(f;x) - f(x)\} = \frac{M_7(r)(-1)^r x^{(r+1)/2} f^{(r+1)}(x)}{(r+1)!}$$

for every $x \in R_0$.

If $r = 0$, then (16) implies (6).

3.3. Finally we shall prove the analogy of (5) for the first derivative of $S_{n,r}(f)$.

Theorem 3. *Suppose that $f \in C_B^r$, $r \in N$. Then*

$$\lim_{n\to\infty} (S_{n,r}(f(t)))'(x) = f'(x) \qquad \text{for } x > 0. \tag{22}$$

P r o o f. The assertion (22) for Szasz-Mirakyan operators $S_n(f)$ and $f \in C_B^1$ is given in [4]. Fix $r \in N$ and $x > 0$. Then by elementary calculations we get from (7) and (3):

$$\frac{d}{dx}S_{n,r}(f;x) = \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r-1} \frac{f^{(j+1)}\left(\frac{k}{n}\right)}{j!}\left(x - \frac{k}{n}\right)^j$$

$$-\frac{n}{x}\sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!}\left(x - \frac{k}{n}\right)^{j+1}, \qquad n \in N.$$

Now $f^{(j)} \in C_B^{r-j}$, $0 \le j \le r$, and as in the proof of Theorem 2 we can write

$$f^{(q)}(t) = \sum_{i=0}^{r-q} \frac{f^{(q+i)}(x)}{i!}(t-x)^i + \psi_q(t;x)(t-x)^{r-q}, \qquad 0 \le q \le r,$$

for $t \in R_0$, where $\psi_q(t)t^{r-q} \equiv \psi_q(t;x)t^{r-q}$ is function belonging to C_B and $\lim_{t\to x}\psi_q(t) = \psi_q(x) = 0$. Consequently we get

$$\frac{d}{dx}S_{n,r}(f;x) = \sum_{k=0}^{\infty} p_k(nx) \sum_{j=0}^{r-1} \frac{(-1)^j}{j!} \sum_{l=j}^{r-1} \frac{f^{(l+1)}(x)}{(l-j)!}\left(\frac{k}{n} - x\right)^l + \tag{23}$$

$$+\sum_{k=0}^{\infty} p_k(nx)\left(\frac{k}{n} - x\right)^{r-1} \sum_{j=0}^{r-1} \frac{(-1)^j}{j!}\psi_{j+1}\left(\frac{k}{n};x\right) +$$

$$+\frac{n}{x}\sum_{k=0}^{\infty}p_k(nx)\sum_{j=0}^{r}\frac{(-1)^j}{j!}\sum_{l=j}^{r}\frac{f^{(l)}(x)}{(l-j)!}\left(\frac{k}{n}-x\right)^{l+1}+$$

$$+\frac{n}{x}\sum_{k=0}^{\infty}p_k(nx)\left(\frac{k}{n}-x\right)^{r+1}\sum_{j=0}^{r}\frac{(-1)^j}{j!}\psi_j\left(\frac{k}{n};x\right):=\sum_{q=1}^{4}Z_{n,q}(x).$$

Arguing as in the proof of Theorem 2 and by (19), we get

$$Z_{n,1}(x)=\sum_{k=0}^{\infty}p_k(nx)\sum_{l=0}^{r-1}\frac{f^{(l+1)}(x)}{l!}\left(\frac{k}{n}-x\right)^{l}\sum_{j=0}^{l}\binom{l}{j}(-1)^j=f'(x), \tag{24}$$

$$Z_{n,3}(x)=\frac{n}{x}\sum_{k=0}^{\infty}p_k(nx)\sum_{l=0}^{r}\frac{f^{(l)}(x)}{l!}\left(\frac{k}{n}-x\right)^{l+1}\sum_{j=0}^{l}\binom{l}{j}(-1)^j= \tag{25}$$

$$=\frac{n}{x}f(x)S_n(t-x;x)=0,$$

$$Z_{n,2}(x)=S_n\left((t-x)^{r-1}\Psi_r(t);x\right),$$

$$Z_{n,4}(x)=\frac{n}{x}S_n\left((t-x)^{r+1}\Psi_r^*(t);x\right),$$

for $n\in N$, where

$$\Psi_r(t)\equiv\Psi_r(t;x):=\sum_{j=0}^{r-1}\frac{(-1)^j}{j!}\psi_{j+1}(t;x),$$

$$\Psi_r^*(t)\equiv\Psi_r^*(t;x):=\sum_{j=0}^{r}\frac{(-1)^j}{j!}\psi_j(t;x).$$

Analogously as for $B_{n,r}(x)$ in the proof of Theorem 2, we can prove that

$$\lim_{n\to\infty}Z_{n,q}(x)=0 \qquad \text{for } q=2,4. \tag{26}$$

Collecting (23)-(26), we immediately obtain (22).

References

[1] M. Becker, D. Kucharski and R. J. Nessel, Global approximation theorems for the Szasz-Mirakjan operators in exponential weight spaces, In: *Linear Spaces and Approximation* (Proc. Conf. Oberwolfach, 1977), Birkhuser Verlag, Basel ISNM, **40** (1978), 319 - 333.

[2] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag, New York 1987.

[3] G. H. Kirov, A generalization of the Bernstein polynomials, *Math. Balkanica*, **6** (1992), 147-153.

[4] L. Rempulska and M. Skorupka, On convergence of first derivatives of certain Szasz-Mirakyan type operators, *Rend. Math. Appl.*, **19** (1999), 269-279.

Institute of Mathematics
Poznań University of Technology
Piotrowo 3A
60-965 Poznań, POLAND

Received 30.09.2003