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Measurability of Sets of Pairs of Skew Nonisotropic Straight Lines in the Simply Isotropic Space I

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The measurable sets of pairs of skew nonisotropic straight lines of type α and the corresponding densities with respect to the group of the general similitudes and some its subgroups are described. Also some Crofton's type formulas are presented.

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1. Introduction

The simply isotropic space $I_3^{(1)}$ is defined as a projective space $\mathbb{P}_3(\mathbb{R})$ in which the absolute consists of a plane ω and two complex conjugate straight lines f_1, f_2 into ω with a real intersection point F [8], [10], [11] . All regular projectivities transforming the absolute figure into itself form the 8-parametric *group* G_8 *of the general simply isotropic similitudes*. Passing on to affine coordinates (x, y, z) a similitude of G_8 can be written in the form [8; p.3]

$$\begin{aligned} x' &= c_1 + c_7(x\cos\varphi - y\sin\varphi),\\ y' &= c_2 + c_7(x\sin\varphi + y\cos\varphi),\\ z' &= c_3 + c_4x + c_5y + c_6z, \end{aligned} \tag{1}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0$ and φ are real parameters.

A straight line is said to be (completely) isotropic if its infinite point coincides with the absolute point F; otherwise the straight line is said to be nonisotropic [8, p.5].

Further, let G_1 and G_2 be two nonisotropic straight lines and denote by U_1 and U_2 their infinite points, respectively. The straight lines G_1 and G_2 are said to be of type α if the points U_1, U_2 and F are noncollinear; otherwise the straight lines are said to be of type β.

We shall consider with G_8 and the following its subgroups:

I . $B_7 \subset G_8 \iff c_7 = 1$. It is the group of the simply isotropic similitudes of the δ-distance [8; p.5].

II . $S_7 \subset G_8 \iff c_6 = 1$. It is the group of the simply isotropic similitudes of the s-distance [8; p.6].

III . $W_7 \subset G_8 \iff c_6 = c_7$. It is the group of the simply isotropic angular similitudes [8; p.18].

IV . $G_7 \subset G_8 \iff \varphi = 0$. It is the group of the boundary simply isotropic similitudes [8; p.8].

V . $V_7 \subset G_8 \iff c_6 {c_7}^2 = 1$. It is the group of the volume preserving simply isotropic similitudes [8; p.8].

VI . $G_6 = G_7 \cap V_7$. It is the group of the volume preserving boundary simply isotropic similitudes [8; p.8].

VII . $B_6 = B_7 \cap G_7$. It is the group of the modular boundary motions [8; p.9].

VIII. $B_5 = B_7 \cap S_7 \cap G_7$. It is the group of the unimodular boundary motions [8; p.9].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [9], G. I. Drinfel'd and A. V. Lucenko [4], [5], [6] we study the measurability of sets of pairs of skew nonisotropic straight lines in ${I_3}^{(1)}$ with respect to G_8 and indicated above subgroups. Analogous problems about sets of pairs of points and pairs of planes in ${I_3}^{(1)}$ have been treated in [1] and [2], respectively.

2. Measurability of sets of pairs of skew nonisotropic straight lines of type alpha

2.1 Measurability with respect to $\mathbf{G_8}$

Let (G_1, G_2) be a pair of skew nonisotropic straight lines of type α determined by the equations

$$(2) \qquad G_i: \quad x = a_i z + p_i, \ y = b_i z + q_i, \qquad i = 1, 2,$$

where

$$(3) \qquad (a_2 - a_1)(q_2 - q_1) - (b_2 - b_1)(p_2 - p_1) \neq 0, \quad a_1 b_2 - a_2 b_1 \neq 0.$$

Under the action of (1) the pair $(G_1, G_2)(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ is transformed into the pair $(G_1', G_2')(a_1', b_1', p_1', q_1', a_2', b_2', p_2', q_2')$ as

$$
\begin{aligned}
a_i' &= K_i c_7 (a_i cos\varphi - b_i sin\varphi), \\
b_i' &= K_i c_7 (a_i sin\varphi + b_i cos\varphi), \\
p_i' &= K_i c_7 \{[-c_3 a_i + c_5(b_i p_i - a_i q_i) + c_6 p_i] cos\varphi + \\
&\qquad + [c_3 b_i + c_4(b_i p_i - a_i q_i) - c_6 q_i] sin\varphi\} + c_1, \\
q_i' &= K_i c_7 \{[-c_3 a_i + c_5(b_i p_i - a_i q_i) + c_6 p_i] sin\varphi - \\
&\qquad - [c_3 b_i + c_4(b_i p_i - a_i q_i) - c_6 q_i] cos\varphi\} + c_2,
\end{aligned} \tag{4}
$$

where $K_i = (c_4 a_i + c_5 b_i + c_6)^{-1}$, $i = 1, 2$.

The transformations (4) form the associated group $\overline{G}_8$ of G_8 [9; p.34]. The group $\overline{G}_8$ is isomorphic to G_8 and the density with respect to G_8 of the pairs (G_1, G_2), if it exists, coincides with the density with respect to $\overline{G}_8$ of the points $(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ in the set of parameters. The associated group $\overline{G}_8$ has the infinitesimal operators

$$Y_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Y_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad Y_3 = a_1\frac{\partial}{\partial p_1} + b_1\frac{\partial}{\partial q_1} + a_2\frac{\partial}{\partial p_2} + b_2\frac{\partial}{\partial q_2},$$

$$Y_4 = a_1(a_1\frac{\partial}{\partial a_1} + b_1\frac{\partial}{\partial b_1}) + p_1(a_1\frac{\partial}{\partial p_1} + b_1\frac{\partial}{\partial q_1}) + a_2(a_2\frac{\partial}{\partial a_2} + b_2\frac{\partial}{\partial b_2}) +$$

$$+ p_2(a_2\frac{\partial}{\partial p_2} + b_2\frac{\partial}{\partial q_2}), \qquad Y_5 = b_1(a_1\frac{\partial}{\partial a_1} + b_1\frac{\partial}{\partial b_1}) + q_1(a_1\frac{\partial}{\partial p_1} + b_1\frac{\partial}{\partial q_1}) +$$

$$b_2(a_2\frac{\partial}{\partial a_2} + b_2\frac{\partial}{\partial b_2}) + q_2(a_2\frac{\partial}{\partial p_2} + b_2\frac{\partial}{\partial q_2}), \; Y_6 = a_1\frac{\partial}{\partial a_1} + b_1\frac{\partial}{\partial b_1} + a_2\frac{\partial}{\partial a_2} + b_2\frac{\partial}{\partial b_2},$$

$$Y_7 = a_1\frac{\partial}{\partial a_1} + b_1\frac{\partial}{\partial b_1} + p_1\frac{\partial}{\partial p_1} + q_1\frac{\partial}{\partial q_1} + a_2\frac{\partial}{\partial a_2} + b_2\frac{\partial}{\partial b_2} + p_2\frac{\partial}{\partial p_2} + q_2\frac{\partial}{\partial q_2},$$

$$Y_8 = b_1\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial b_1} + q_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial q_1} + b_2\frac{\partial}{\partial a_2} - a_2\frac{\partial}{\partial b_2} + q_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial q_2}.$$

The group $\overline{G}_8$ acts intransitively on the set of points $(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ and therefore the set of pairs (G_1, G_2) has not invariant density under G_8. The system

$$Y_i(f) = 0, \qquad i = 1, 2, ..., 8 \tag{5}$$

has the solution

$$f = \frac{a_1 a_2 + b_1 b_2}{a_1 b_2 - a_2 b_1} \tag{6}$$

and it is an absolute invariant of $\overline{G}_8$. Consider the subset of pairs (G_1, G_2) of skew nonisotropic straight lines of type α satisfying the condition

$$\frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1} = h, \tag{7}$$

where $h = const$. The group $\overline{G}_8$ induces on the subset (7) the group $G_8^\star$ with the infinitesimal operators

$$Z_1 = \tfrac{\partial}{\partial p_1} + \tfrac{\partial}{\partial p_2},\ Z_2 = \tfrac{\partial}{\partial q_1} + \tfrac{\partial}{\partial q_2},\ Z_3 = a_1\tfrac{\partial}{\partial p_1} + b_1\tfrac{\partial}{\partial q_1} + a_2\tfrac{\partial}{\partial p_2} + \tfrac{a_2(a_1+hb_1)}{ha_1-b_1}\tfrac{\partial}{\partial q_2},$$

$$Z_4 = a_1(a_1\tfrac{\partial}{\partial a_1} + b_1\tfrac{\partial}{\partial b_1}) + p_1(a_1\tfrac{\partial}{\partial p_1} + b_1\tfrac{\partial}{\partial q_1}) + a_2^2\tfrac{\partial}{\partial a_2} + a_2p_2(\tfrac{\partial}{\partial p_2} + \tfrac{a_1+hb_1}{ha_1-b_1}\tfrac{\partial}{\partial q_2}),$$

$$Z_5 = b_1(a_1\tfrac{\partial}{\partial a_1} + b_1\tfrac{\partial}{\partial b_1}) + q_1(a_1\tfrac{\partial}{\partial p_1} + b_1\tfrac{\partial}{\partial q_1}) + a_2^2\tfrac{a_1+hb_1}{ha_1-b_1}\tfrac{\partial}{\partial a_2}$$

$$+ a_2q_2(\tfrac{\partial}{\partial p_2} + \tfrac{a_1+hb_1}{ha_1-b_1}\tfrac{\partial}{\partial q_2}), \quad Z_6 = a_1\tfrac{\partial}{\partial a_1} + b_1\tfrac{\partial}{\partial b_1} + a_2\tfrac{\partial}{\partial a_2},$$

$$Z_7 = a_1\tfrac{\partial}{\partial a_1} + b_1\tfrac{\partial}{\partial b_1} + p_1\tfrac{\partial}{\partial p_1} + q_1\tfrac{\partial}{\partial q_1} + a_2\tfrac{\partial}{\partial a_2} + p_2\tfrac{\partial}{\partial p_2} + q_2\tfrac{\partial}{\partial q_2},$$

$$Z_8 = b_1\tfrac{\partial}{\partial a_1} - a_1\tfrac{\partial}{\partial b_1} + q_1\tfrac{\partial}{\partial p_1} - p_1\tfrac{\partial}{\partial q_1} + \tfrac{a_2(a_1+hb_1)}{ha_1-b_1}\tfrac{\partial}{\partial a_2} + q_2\tfrac{\partial}{\partial p_2} - p_2\tfrac{\partial}{\partial q_2}.$$

From $|a_1 - a_2| + |b_1 - b_2| \neq 0$ it follows that at least one of the diferences $a_1 - a_2$ and $b_1 - b_2$ is not zero. We can assume without loss of generality that $a_1 - a_2 \neq 0$. In this case the infinitesimal operators $Z_1, Z_2, Z_3, Z_4, Z_6, Z_7$ and Z_8 are arcwise unconnected and

$$Z_5 = \lambda_1 Z_1 + \lambda_2 Z_2 + \lambda_3 Z_3 + \lambda_4 Z_4 + \lambda_7 Z_7,$$

where

$$\lambda_1 = \frac{a_2b_1p_1 - a_1b_2p_2 + a_1a_2(q_2 - q_1)}{a_2 - a_1},\ \lambda_2 = \frac{-a_1b_2q_1 + a_2b_1q_2 - b_1b_2(p_2 - p_1)}{a_2 - a_1},$$

$$\lambda_3 = \frac{b_1p_1 - b_2p_2 - a_1q_1 + a_2q_2}{a_2 - a_1}, \quad \lambda_4 = \frac{b_2 - b_1}{a_2 - a_1}, \quad \lambda_7 = \frac{-a_2b_1 + a_1b_2}{a_2 - a_1}.$$

Since

$$Z_1(\lambda_1) + Z_2(\lambda_2) + Z_3(\lambda_3) + Z_4(\lambda_4) + Z_7(\lambda_7) \neq 0,$$

it follows that a set of pairs (G_1, G_2) satisfying the condition (7) is not measurable under G_8 and has not measurable subsets. Thus we establish the following result:

Theorem 2.1.1 *Sets of pairs of skew nonisotropic straight lines of type α is not measurable with respect to G_8 and has not measurable subsets.*

2.2 Some Crofton's type formulas under G_8

The integral (6) of the system (5) is an absolute invariant of the set of the pairs (G_1, G_2) with respect to the group G_8. Then we can introduce a density for the set of the pairs $(G_1, G_2)(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ of skew nonisotropic straight lines of type α under G_8 by the equality

$$(8) \quad d(G_1, G_2) = \left|\frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1}\right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge db_2 \wedge dp_2 \wedge dq_2.$$

Remark 2.2.1 *We note that the quantity ψ determined by*

$$(9) \qquad sin\psi = \frac{a_1b_2 - a_2b_1}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}, \qquad cos\psi = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}},$$

is an absolute invariant of the pairs (G_1, G_2) with respect to the group $B_6^{(1)}$ of the simply isotropic motions in $I_3^{(1)}$ and it is called the angle from G_1 to G_2 [9; p.45].

Replacing (9) into (8) we find another expression for the density:

$$(10) \qquad d(G_1, G_2) = |cotg\psi| \; da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge db_2 \wedge dp_2 \wedge dq_2.$$

Let φ_i be the angle of the straight line G_i with the horizontal plane Oxy. Then [9; p.48]

$$(11) \qquad \varphi_i = \frac{1}{\sqrt{a_i^2 + b_i^2}}.$$

Let ι_1 and ι_2 be the isotropic planes [8; p.16] through G_1 and G_2, respectively, and denote $J_i^1 = \iota_i \cap Oxz$, $J_i^2 = \iota_i \cap Oyz$, $i = 1, 2$. If χ_i^1, χ_i^2 are the angles between G_i and Oxz, Oyz, respectively, we have [8; p.48]

$$(12) \qquad sin\chi_i^1 = \frac{b_i}{\sqrt{a_i^2 + b_i^2}}, \qquad sin\chi_i^2 = \frac{a_i}{\sqrt{a_i^2 + b_i^2}}$$

and therefore

$$(13) \qquad d\chi_i^1 = -d\chi_i^2 = -\frac{b_i da_i - a_i db_i}{a_i^2 + b_i^2}$$

On the other hand, the straight lines J_i^1 and J_i^2 are determined by the equations

$$J_i^1: \quad x = \alpha_i,\ y = 0,$$
$$J_i^2: \quad y = \beta_i,\ x = 0,$$

where

$$\alpha_i = p_i - \frac{a_i q_i}{b_i}, \qquad \beta_i = q_i - \frac{b_i p_i}{a_i}.$$

Then

$$d(J_1^1, J_2^1) = \frac{d\alpha_1 \wedge d\alpha_2}{(\alpha_2 - \alpha_1)^2} \tag{14}$$

and

$$d(J_1^2, J_2^2) = \frac{d\beta_1 \wedge d\beta_2}{(\beta_2 - \beta_1)^2} \tag{15}$$

are the densities for the pairs (J_1^1, J_2^1) and (J_1^2, J_2^2) with respect to the groups of the general similitudes in the isotropic planes Oxz and Oyz, respectively.

Putting (11), (12), (13), (14) and (15) in (10), we find

$$d(G_1, G_2) = \left| \frac{cotg\psi A^2 B^2}{\varphi_1 \varphi_2 sin\chi_1^1 sin\chi_2^1} \right| d(J_1^1, J_2^1) \wedge d(J_1^2, J_2^2) \wedge d\chi_1^1 \wedge d\chi_2^1 \wedge d\varphi_1 \wedge d\varphi_2 \tag{16}$$

and

$$d(G_1, G_2) = \left| \frac{cotg\psi A^2 B^2}{\varphi_1 \varphi_2 sin\chi_1^1 sin\chi_2^1} \right| d(J_1^1, J_2^1) \wedge d(J_1^2, J_2^2) \wedge d\chi_1^2 \wedge d\chi_2^2 \wedge d\varphi_1 \wedge d\varphi_2, \tag{17}$$

where

$$\begin{aligned} A =& \quad a_1(a_2 q_2 - b_2 p_2) - a_2(a_1 q_1 - b_1 p_1), \\ B =& \ -b_1(a_2 q_2 - b_2 p_2) + b_2(a_1 q_1 - b_1 p_1). \end{aligned}$$

Further, if we denote $Q_i^1 = G_i \cap Oxz$, $Q_i^2 = G_i \cap Oyz$, $i = 1, 2$, then we have

$$Q_1^1(-\tfrac{a_1 q_1}{b_1} + p_1, 0, -\tfrac{q_1}{b_1}), \quad Q_2^1(-\tfrac{a_2 q_2}{b_2} + p_2, 0, -\tfrac{q_2}{b_2}),$$

$$Q_1^2(0, -\tfrac{b_1 p_1}{a_1} + q_1, -\tfrac{p_1}{a_1}), \quad Q_2^2(0, -\tfrac{b_2 p_2}{a_2} + q_2, -\tfrac{p_2}{a_2}).$$

Hence

$$\delta^1 = \delta(Q_1^1, Q_2^1) = \frac{B}{b_1 b_2}, \qquad \delta^2 = \delta(Q_1^2, Q_2^2) = \frac{A}{a_1 a_2} \tag{18}$$

are the corresponding isotropic distances [7; p.12]. Replacing (18) to (16) and (17) we obtain

$$(19)\quad d(G_1,G_2)=\left|\frac{cotg\psi\ \delta^1\delta^2}{\varphi_1^3\varphi_2^3}\right|\ d(J_1^1,J_2^1)\wedge d(J_1^2,J_2^2)\wedge d\chi_1^1\wedge d\chi_2^1\wedge d\varphi_1\wedge d\varphi_2$$

and

$$(20)\quad d(G_1,G_2)=\left|\frac{cotg\psi\ \delta^1\delta^2}{\varphi_1^3\varphi_2^3}\right|\ d(J_1^1,J_2^1)\wedge d(J_1^2,J_2^2)\wedge d\chi_1^2\wedge d\chi_2^2\wedge d\varphi_1\wedge d\varphi_2.$$

We can summarize the foregoing results in the following

Theorem 2.2.1 *The density (8) for the pairs (G_1, G_2) of skew nonisotropic straight lines of type α, determined by (2) and (3) can be written in the forms (10), (16), (17), (19) and (20).*

If we project the straight lines G_1 and G_2 orthogonally on Oxy, we obtain Crofton's type formulas like to the ones in [3].

2.3 Measurability with respect to the groups $\mathbf{B_7}, \mathbf{S_7}, \mathbf{W_7}, \mathbf{G_7}, \mathbf{V_7}, \mathbf{G_6}, \mathbf{B_6}$ and $\mathbf{B_5}$

By arguments similar to the ones used in the section 2.1 we establish the measurability with respect to all the rest groups. We have the following results:

Theorem 2.3.1 *With respect to the groups B_7, W_7 and V_7 the set of pairs (G_1, G_2) of skew nonisotropic straight lines of type α is not measurable with respect to B_7 but it has the measurable subset*

$$\frac{a_1a_2+b_1b_2}{a_2b_1-a_1b_2}=h,\qquad h=const$$

with the density

$$d(G_1,G_2)=$$
$$=\frac{|a_2|(a_1^2+b_1^2)(ha_1+b_1)^2da_1\wedge db_1\wedge dp_1\wedge dq_1\wedge da_2\wedge dp_2\wedge dq_2}{[(a_2b_1h-a_1a_2-b_1^2-a_1b_1h)(p_2-p_1)-(a_2-a_1)(a_1h+b_1)(q_2-q_1)]^4}.$$

Theorem 2.3.2 *With respect to the group S_7 the set of pairs (G_1, G_2) of skew nonisotropic straight lines of type α is not measurable but it has the measurable subset*

$$\frac{a_1a_2+b_1b_2}{a_2b_1-a_1b_2}=h_1,\qquad \frac{(a_2-a_1)(q_2-q_1)-(b_2-b_1)(p_2-p_1)}{a_2b_1-a_1b_2}=h_2,$$

where $h_1 = const$, $h_2 = const$, *with the density*

$$d(G_1, G_2) = \left| \frac{a_1 h_1 + b_1}{(a_2 - a_1) a_2^2 (a_1^2 + b_1^2)} \right| \, da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2.$$

Theorem 2.3.3 *With respect to the groups* G_7, G_6 *and* B_6 *the set of pairs* (G_1, G_2) *of skew nonisotropic straight lines of type* α *is not measurable but it has the measurable subset*

$$\frac{b_1}{a_1} = h_1, \qquad \frac{b_2}{a_2} = h_2,$$

where $h_1 = const$, $h_2 = const$, *with the density*

$$d(G_1, G_2) =$$

$$= \frac{|a_1 a_2| da_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2}{[(a_2 - a_1)(q_2 - q_1) - (a_2 h_2 - a_1 h_1)(p_2 - p_1)]^4}.$$

Theorem 2.3.4 *With respect to the group* B_5 *the set of pairs* (G_1, G_2) *of skew nonisotropic straight lines of type* α *is not measurable but it has the measurable subset*

$$\frac{b_1}{a_1} = h_1, \ \frac{b_2}{a_2} = h_2, \ (b_2 - b_1)(p_2 - p_1) - (a_2 - a_1)(q_2 - q_1) = h_3,$$

where $h_1 = const$, $h_2 = const$, $h_3 = const$, *with the density*

$$d(G_1, G_2) = \left| \frac{1}{a_1^2 a_2^2 (a_2 - a_1)} \right| \, da_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2.$$

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