Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 1-2

Measurability of Sets of Pairs of Skew Nonisotropic Straight Lines in the Simply Isotropic Space II

Adrijan V. Borisov[1] , *Margarita G. Spirova*[2]

The measurable sets of pairs of skew nonisotropic straight lines of type β and the corresponding densities with respect to the group of the general similitudes and some its subgroups are described. Also some Crofton type formulas are presented. Analogous problems about sets of pairs of nonisotropic straight lines of type α have been treated in [13].

AMS Subj. Classification: 53C65

Key Words: simply isotropic space, measurability, density

3. Measurability of sets of pairs of skew nonisotropic straight lines of type beta

3.1 Measurability with respect to $\mathbf{G_8}$

Now let us consider a pair of skew nonisotropic straight lines of type β (G_1, G_2) determined by the equations

$$\begin{aligned} G_1 &: x = a_1 z + p_1, \qquad y = b_1 z + q_1, \qquad a_1 \neq 0, \\ G_2 &: x = a_2 z + p_2, \qquad y = \tfrac{a_2}{a_1}\, b_1 z + q_2, \qquad a_2 \neq 0. \end{aligned} \tag{21}$$

Under the action of transformations (1) from [13] the pair

$$(G_1, G_2)\,(a_1, b_1, p_1, q_1, a_2, p_2, q_2)$$

is transformed into the pair $(G_1{}', G_2{}')\,(a_1{}', b_1{}', p_1{}', q_1{}', a_2{}', p_2{}', q_2{}')$ as

$$
\begin{aligned}
&a_1' = K_1c_7(a_1cos\varphi - b_1sin\varphi),\\
&b_1' = K_1c_7(a_1sin\varphi + b_1cos\varphi),\\
&p_1' = K_1c_7\{[-c_3a_1 + c_5(b_1p_1 - a_1q_1) + c_6p_1]cos\varphi +\\
&\qquad +[c_3b_1 + c_4(b_1p_1 - a_1q_1) - c_6q_1]sin\varphi\} + c_1,\\
&q_1' = K_1c_7\{[-c_3a_1 + c_5(b_1p_1 - a_1q_1) + c_6p_1]sin\varphi -\\
&\qquad -[c_3b_1 + c_4(b_1p_1 - a_1q_1) - c_6q_1]cos\varphi\} + c_2,\\
&a_2' = K_2c_7(a_2cos\varphi - \tfrac{a_2b_1}{a_1}sin\varphi),\\
&p_2' = K_2c_7\{[-c_3a_2 + c_5(\tfrac{a_2b_1}{a_1}p_2 - a_2q_2) + c_6p_2]cos\varphi +\\
&\qquad +[c_3\tfrac{a_2b_1}{a_1} + c_4(\tfrac{a_2b_1}{a_2}p_2 - a_2q_2) - c_6q_2]sin\varphi\} + c_1,\\
&q_2' = K_2c_7\{[-c_3a_2 + c_5(\tfrac{a_2b_1}{a_1}p_2 - a_2q_2) + c_6p_2]sin\varphi -\\
&\qquad -[c_3\tfrac{a_2b_1}{a_1} + c_4(\tfrac{a_2b_1}{a_1}p_2 - a_2q_2) - c_6q_2]cos\varphi\} + c_2,
\end{aligned}
\tag{22}
$$

where $K_1 = (c_4a_1 + c_5b_1 + c_6)^{-1}$ and $K_2 = (c_4a_2 + c_5\frac{a_2b_1}{a_1} + c_6)^{-1}$.

The transformations (22) form the associated group $\overline{G}_8$ of G_8 [9; p.34] which has the infinitesimal operators

$$Y_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Y_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad Y_3 = a_1\frac{\partial}{\partial p_1} + b_1\frac{\partial}{\partial q_1} + a_2\frac{\partial}{\partial p_2} + \frac{a_2b_1}{a_1}\frac{\partial}{\partial q_2},$$

$$Y_4 = a_1^2\frac{\partial}{\partial a_1} + a_1b_1\frac{\partial}{\partial b_1} + a_1p_1\frac{\partial}{\partial p_1} + b_1p_1\frac{\partial}{\partial q_1} + a_2^2\frac{\partial}{\partial a_2} + a_2p_2\frac{\partial}{\partial p_2} + \frac{a_2b_1}{a_1}p_2\frac{\partial}{\partial q_2},$$

$$Y_5 = a_1b_1\frac{\partial}{\partial a_1} + b_1^2\frac{\partial}{\partial b_1} + a_1q_1\frac{\partial}{\partial p_1} + b_1q_1\frac{\partial}{\partial q_1} + \frac{a_2^2b_1}{a_1}\frac{\partial}{\partial a_2} + a_2q_2\frac{\partial}{\partial p_2} + \frac{a_2b_1}{a_1}q_2\frac{\partial}{\partial q_2},$$

$$Y_6 = a_1\frac{\partial}{\partial a_1} + b_1\frac{\partial}{\partial b_1} + a_2\frac{\partial}{\partial a_2},$$

$$Y_7 = a_1\frac{\partial}{\partial a_1} + b_1\frac{\partial}{\partial b_1} + p_1\frac{\partial}{\partial p_1} + q_1\frac{\partial}{\partial q_1} + a_2\frac{\partial}{\partial a_2} + p_2\frac{\partial}{\partial p_2} + q_2\frac{\partial}{\partial q_2},$$

$$Y_8 = b_1\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial b_1} + q_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial q_1} + \frac{a_2b_1}{a_1}\frac{\partial}{\partial a_2} + q_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial q_2}$$

and it acts transitively on the set of parameters $(a_1, b_1, p_1, q_1, a_2, p_2, q_2)$. The integral invariant function $f = f(a_1, b_1, p_1, q_1, a_2, p_2, q_2)$ satisfies the system of

R. Deltheil [14, p.28], [9, p.11]

$$Y_1(f)=0, \qquad Y_2(f)=0, \qquad Y_3(f)=0, \qquad Y_4(f)+(4a_1+3a_2)f=0,$$

$$Y_5(f)+(4b_1+3\tfrac{a_2b_1}{a_1})f=0,\ Y_6(f)+3f=0,\ Y_7(f)+7f=0,\ Y_8(f)+\tfrac{b_1}{a_1}f=0.$$

and has the form

$$f=\frac{h\, a_1^2a_2(a_1^2+b_1^2)}{(a_1-a_2)^4[b_1(p_2-p_1)-a_1(q_2-q_1)]^4},$$

where $h=const.$

So we state the following

Theorem 3.1.1 *The set of pairs (G_1,G_2) of skew nonisotropic straight lines of type β determined by (21) is measurable with respect to the group G_8 and has the density*

(23) $$d(G_1,G_2)=$$

$$=\left|\frac{a_1^2a_2(a_1^2+b_1^2)}{(a_1-a_2)^4[b_1(p_2-p_1)-a_1(q_2-q_1)]^4}\right| da_1\wedge db_1\wedge dp_1\wedge dq_1\wedge da_2\wedge dp_2\wedge dq_2.$$

3.2 Some Crofton's type formulas under $\mathbf{G_8}$

Note that [8; p.45]

(24) $$a=\frac{b_1(p_2-p_1)-a_1(q_2-q_1)}{\sqrt{a_1^2+b_1^2}}$$

and

(25) $$s=\frac{a_1-a_2}{a_2\sqrt{a_1^2+b_1^2}}$$

are the distance and the angle from G_1 to G_2, respectively, from (23) we obtain

(26) $$d(G_1,G_2)=\left|\frac{a_1^2}{s^4a^4a_2^3(a_1^2+b_1^2)^3}\right| da_1\wedge db_1\wedge dp_1\wedge dq_1\wedge da_2\wedge dp_2\wedge dq_2.$$

Let $\widetilde{G_i}$ be the orthogonal projections of G_i into Oxy (i=1, 2). Then

$$\widetilde{G_i}:\quad x-\frac{a_1}{b_1}\,y+\frac{a_1q_i}{b_1}-p_i=0,\ \ z=0$$

and

$$d(\widetilde{G_1}, \widetilde{G_2}) =$$

$$\frac{1}{(a_1^2+b_1^2)[b_1(p_2-p_1)-a_1(q_2-q_1)]^2} (b_1^5\, dp_1 \wedge dp_2 \wedge da_1 - a_1 b_1^4\, dp_1 \wedge dq_2 \wedge da_1$$

$$(27) \quad -a_1 b_1^4\, dq_1 \wedge dp_2 \wedge da_1 + a_1^2 b_1^3\, dq_1 \wedge dq_2 \wedge da_1 - a_1 b_1^4\, dp_1 \wedge dp_2 \wedge db_1$$

$$+a_1^2 b_1^3\, dp_1 \wedge dq_2 \wedge db_1 + a_1^2 b_1^3\, dq_1 \wedge dp_2 \wedge db_1 - a_1^3 b_1^2\, dq_1 \wedge dq_2 \wedge db_1)$$

is the density for the pairs $(\widetilde{G_1}, \widetilde{G_2})$ in Oxy. We note that the plane Oxy is Euclidean and $d(\widetilde{G_1}, \widetilde{G_2})$ is the density for pairs of parallel straight lines under the group of the similitudes . By differentiation of (24) and (25) and by serial exterior multiplication of (27) and $dp_1 \wedge p_2$, $dp_1 \wedge q_2$, $dp_2 \wedge q_1$, $dq_1 \wedge q_2$ we get

$$\begin{aligned} d(G_1, G_2) &= \left| \frac{\sqrt{a_1^2+b_1^2}}{a_1^2 b_1^3 a_2 s^4 a^3} \right| da \wedge ds \wedge dp_1 \wedge dp_2 \wedge d(\widetilde{G_1}, \widetilde{G_2}) = \\ &= \left| \frac{\sqrt{a_1^2+b_1^2}}{a_1 b_1^4 a_2 s^4 a^3} \right| da \wedge ds \wedge dp_1 \wedge dq_2 \wedge d(\widetilde{G_1}, \widetilde{G_2}) = \\ &= \left| \frac{\sqrt{a_1^2+b_1^2}}{a_1 b_1^4 a_2 s^4 a^3} \right| da \wedge ds \wedge dq_1 \wedge dp_2 \wedge d(\widetilde{G_1}, \widetilde{G_2}) = \\ &= \left| \frac{\sqrt{a_1^2+b_1^2}}{b_1^5 a_2 s^4 a^3} \right| da \wedge ds \wedge dq_1 \wedge dq_2 \wedge d(\widetilde{G_1}, \widetilde{G_2}). \end{aligned} \tag{28}$$

Thus we establish the following

Theorem 3.2.1 *The density for the set of pairs (G_1, G_2) of skew non-isotropic straight lines of type β with respect to the group G_8 satisfies the relations (26) and (28).*

3.3 Measurability with respect to B_7

Similarly to section 3.1 we establish the following

Theorem 3.3.1 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group B_7 but it has the measurable subset*

$$\frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}} = h$$

with the density

$$d(G_1, G_2) = \left| \frac{a_1 a_2}{(a_1 - a_2)^4 \sqrt{a_1^2 + a_2^2}} \right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2.$$

3.4 Some Crofton's type formulas under $\mathbf{B_7}$

It is easy to verify that

$$f = \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}}$$

is an absolute invariant of the pairs (G_1, G_2) under B_7. For that reason we can define the density for the set of pairs (21) by the equality

(29)

$$d(G_1, G_2) = \left| \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}} \right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2.$$

By (24) we can write (29) in the form

(30) $$d(G_1, G_2) = |a| \, da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2.$$

Let G_i^1 and G_i^2 be the projections of G_i into Oxz and Oyz obtained in a parallel way to Oy and Ox, respectively. Then

$$G_i^1: \quad x - a_i z - p_i = 0, \qquad y = 0, \qquad i = 1, 2,$$

$$G_1^2: \quad y - b_1 z - q_1 = 0, \qquad x = 0$$

$$G_2^2: \quad y - \tfrac{a_2}{a_1} b_1 z - q_2 = 0, \quad x = 0$$

and

(31) $$d(G_1^1, G_2^1) = \left| \frac{1}{a_1 a_2 (\psi^1)^4} \right| da_1 \wedge dp_1 \wedge da_2 \wedge dp_2$$

is the density for the pairs (G_1^1, G_2^1) in the isotropic plane Oxz under the group of similitudes of the second type [7, p.14], [12, p.189], where

$$\psi^1 = \frac{1}{a_2} - \frac{1}{a_1}$$

is the angle from G_1^1 to G_2^1.

If we denote $Q_i^1 = G_i \cap Oxz$, $Q_i^2 = G_i \cap Oyz$, $i = 1, 2$, then we have

$$Q_1^1(-\tfrac{a_1 q_1}{b_1} + p_1, 0, -\tfrac{q_1}{b_1}), \quad Q_2^1(-\tfrac{a_1 q_2}{b_1} + p_2, 0, -\tfrac{a_1 q_2}{b_1 a_2}),$$

$$Q_1^2(0, -\tfrac{b_1 p_1}{a_1} + q_1, -\tfrac{p_1}{a_1}), \quad Q_2^2(0, -\tfrac{b_1 p_2}{a_1} + q_2, -\tfrac{p_2}{a_2})$$

and

$$(32) \qquad \begin{aligned} \delta^1 &= \delta^1(Q_1^1, Q_2^1) = \tfrac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{b_1}, \\ \delta^2 &= \delta^2(Q_1^2, Q_2^2) = \tfrac{a_1(q_2 - q_1) - b_1(p_2 - p_1)}{a_1} \end{aligned}$$

are the corresponding isotropic distances [7, p.12]. By differentiation of (32) and by exterior multiplication we get

$$(33) \qquad d\delta^1 \wedge d\delta^2 = \frac{a_1}{b_1}\, dq_1 \wedge dq_2.$$

Analogously,

$$\psi^2 = \frac{a_1}{a_2 b_1} - \frac{1}{b_1}$$

is the angle from G_1^2 to G_2^2 and consequently

$$(34) \qquad d\psi^2 = \frac{1}{a_2 b_1}\, da_1 - \frac{a_1 - a_2}{a_2 b_1^2}\, db_1 - \frac{a_1}{a_2^2 b_1}\, da_2.$$

Replacing (31), (33) and (34) into (30) we obtain

$$(35) \qquad d(G_1, G_2) = \left| \frac{a\, b_1^3 a_2 (\psi^1)^3}{a_1} \right| d(G_1^1, G_2^1) \wedge d\psi^2 \wedge d\delta^1 \wedge d\delta^2.$$

Thus we establish the following

Theorem 3.4.1 *The density for the set of pairs (G_1, G_2) of skew non-isotropic straight lines of type β with respect to the group B_7 satisfies the relations (30) and (35).*

3.5 Measurability with respect to the groups $\mathbf{S_7}, \mathbf{W_7}, \mathbf{G_7}, \mathbf{V_7}, \mathbf{G_6}$, $\mathbf{B_6}$ and $\mathbf{B_5}$

By arguments similar to the used in the section 2.1 we find:

Theorem 3.5.1 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group B_7 but it has the measurable subset*

$$\frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + a_2^2}} = h, \quad h = const$$

with the density

$$d(G_1, G_2) = \left| \frac{a_1 a_2}{(a_1 - a_2)^4 \sqrt{a_1^2 + a_2^2}} \right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2.$$

Theorem 3.5.2 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group S_7 but it has the measurable subset*

$$\frac{(a_1 - a_2)[b_1(p_2 - p_1) - a_1(q_2 - q_1)]}{a_2(a_1^2 + b_1^2)} = h, \quad h = const$$

with the density

$$d(G_1, G_2) = \left| \frac{a_1}{a_2^2(a_1 - a_2)(a_1^2 + b_1^2)^2} \right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2.$$

Theorem 3.5.3 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group W_7 but it has the measurable subset*

$$\frac{(a_1 - a_2)}{a_2\sqrt{a_1^2 + b_1^2}} = h, \quad h = const$$

with the density

$$d(G_1, G_2) =$$

$$= \left| \frac{1 + h\sqrt{a_1^2 + b_1^2}}{a_2^2\sqrt{a_1^2 + b_1^2}[b_1(p_2 - p_1) - a_1(q_2 - q_1)]^4} \right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.$$

Theorem 3.5.4 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group G_7 but it has the measurable subset*

$$\frac{b_1}{a_1} = h, \quad h = const$$

with the density

$$d(G_1, G_2) =$$
$$= \left| \frac{a_1^5}{a_2(a_1 - a_2)^4 [b_1(p_2 - p_1) - a_1(q_2 - q_1)]^4} \right| da_1 \wedge dp_1 \wedge da_2 \wedge dq_1 \wedge dp_2 \wedge dq_2.$$

Theorem 3.5.5 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group V_7 but it has the measurable subset*

$$\frac{(a_1 - a_2)[b_1(p_2 - p_1) - a_1(q_2 - q_1)]^3}{a_2(a_1^2 + b_1^2)^2} = h, \qquad h = const$$

with the density

$$d(G_1, G_2) =$$
$$= \left| \frac{a_1^6 (1 - h\sqrt{a_1^2 + b_1^2}\,)^2 [b_1(p_2 - p_1) - a_1(q_2 - q_1)]^3}{(a_1^2 + b_1^2)^{\frac{11}{2}}} \right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.$$

Theorem 3.5.6 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group G_6 but it has the measurable subset*

$$\frac{b_1}{a_1} = h_1, \quad \frac{a_1^2\, a_2^3\, [b_1(p_2 - p_1) - a_1(q_2 - q_1)]}{(a_1 - a_2)^3} = h_2, \qquad h_1, h_2 = const$$

with the density

$$d(G_1, G_2) = \left| \frac{1}{(a_1 - a_2)^3} \right| da_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2.$$

Theorem 3.5.7 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group B_6 but it has the measurable subset*

$$\frac{b_1}{a_1} = h_1, \quad \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{a_1} = h_2, \qquad h_1, h_2 = const$$

with the density

$$d(G_1, G_2) = \left| \frac{1}{(a_1 - a_2)^3} \right| da_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2.$$

Theorem 3.5.8 *The set of pairs (G_1, G_2) of skew nonisotropic straight lines of type β is not measurable with respect to the group G_6 but it has the measurable subset*

$$\frac{b_1}{a_1} = h_1, \quad \frac{a_1 - a_2}{a_1 a_2} = h_2, \quad \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{a_1} = h_3, \qquad h_1, h_2, h_3 = const$$

with the density

$$d(G_1, G_2) = \left| \frac{1}{a_1 a_2^3} \right| \, da_1 \wedge dp_1 \wedge dq_1 \wedge dp_2.$$

References

[12] A. B o r i s o v. Integral geometry in the Galilean plane, *Research work qualifying for a degree full-professor,* Sofia, 1998.

[13] A. B o r i s o v and M. S p i r o v a. Measurability of sets of pairs of skew nonisotropic straight lines in the simply isotropic space I (to appear).

[14] R. D e l t h e i l. Sur la theorie des probabilité géométriques, *Thése Ann. Fac. Sc. Univ. Toulouse*, **11** (1919), (3), 1-65.

[1] *Dept. of Descriptive Geometry*
Univ. of Architecture Civil Eng. and Geodesy
Sofia 1046, BULGARIA
E-MAIL: adribor_fgs@uacg.bg

Received 30.09.2003

[2] *Dept. of Algebra and Geometry*
Shumen University "Ep. K. Preslavski"
Shumen 9712, BULGARIA
E-MAIL: margspr@fmi.shu-bg.net