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Upper Semicontinuous Functional Differential Inclusions

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In this paper we pose an appropriate extension of the so called one sided Lipschitz condition in case of discontinuous functional differential inclusion. The condition here is called strong one sided Lipschitz and we extend the most important properties of the one sided Lipschitz ordinary differential inclusions to the case of strong one sided Lipschitz functional differential inclusions. We obtain also an exponential formula for the reachable set.

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1. Introduction

The aim of the present paper is to impose an appropriate extension of the so called one Sided Lipschitz (OSL) condition in case of discontinuous functional-differential inclusion (FDI):

$$\dot{x}(t) \in F(t, x_t),\ x_0 = \varphi \in \mathbb{B},\ t \in J = [0, 1].$$

Here $F : J \times \mathbb{B} \rightrightarrows E$ with $E \equiv \mathbb{R}^n$, $\mathbb{B} = C([-\tau, 0], E)$ is the space of the continuous functions equipped with the usual max norm $\|\cdot\|_{\mathbb{B}}$ and $x_t \in \mathbb{B}$ is defined by $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$. Further $\rightrightarrows$ is used for multi-valued maps, all the multifunctions are assumed to be with nonempty compact values.

The OSL condition is defined first in case of functional - differential equations in [12]. More general dissipative type conditions are studied in [11] for ordinary differential equations. This condition is extended in case of (jointly) continuous multifunction $F(\cdot, \cdot)$ in [5]. When F is compact valued and almost continuous this condition is studied also in [6]. In case of $\mathbb{R}^n$ and almost continuous $F(\cdot, \cdot)$ the Euler discrete approximations under OSL condition are studied in [8] in ordinary and functional differential cases. We refer to [9] for the theory of functional differential equations. Functional differential inclusions are studied in [1, 8].

We recall some definitions and notations and refer to [2, 10] for all concepts used in this paper but not explicitly discussed.

We say that the multifunction between (complete) metric spaces $H : \mathcal{M} \rightrightarrows \mathcal{N}$ is upper semicontinuous (USC) (respectively lower semicontinuous (LSC), continuous) at $x \in \mathcal{M}$ when for every $\varepsilon > 0$ there exits $\delta > 0$ such that $H(U_\delta(x_0)) \subset V_\varepsilon(H(x_0))$ (for every $h_0 \in H(x_0)$ and every $x_i \to x_0$ there exist $h_i \in H(x_i)$ with $h_i \to h_0$, H is continuous with respect to the Hausdorff metric), where $U_\delta(\cdot)$ and $V_\varepsilon(\cdot)$ are open δ and ε neighborhoods of x_0 and $H(x_0)$. The multimap $G : [0,1] \times E \rightrightarrows E$ is said to be almost USC (LSC) when for any $\varepsilon > 0$ there exists $I_\varepsilon \subset I$ with Lebesgue measure $\text{meas}([0,1] \setminus I_\varepsilon) < \varepsilon$ such that the restriction of G on $I_\varepsilon \times E$ is USC (LSC). The almost continuous multimaps are defined analogously. The support function of the bounded set A is denoted by $\sigma(y, A) := \sup\limits_{x \in A} \langle x, y \rangle$ and the Hausdorff distance between the (closed bounded) sets A, C by $D_H(A, C) = \max\{ex(A, C), ex(C, A)\}$, where $ex(A, C) = \sup\limits_{a \in A} \inf\limits_{b \in C} |a - b|$. By $U_{\mathbb{B}}$ we denote the open unit ball in $\mathbb{B}$, by $\bar{A}$ the closure of A and by $\overline{co}\ A$ the closed convex hull of A.

Definition 1.1. The mapping $F : J \times \mathbb{B} \rightrightarrows E$ is said to be one sided Lipschitz (OSL) when:

There exists a constant L such that

$$\sigma(\alpha(0) - \beta(0), F(t, \alpha)) - \sigma(\alpha(0) - \beta(0), F(t, \beta)) \le L|\alpha(0) - \beta(0)|^2$$

for every $\alpha - \beta \in \mathbb{B}_0 := \{\chi \in \mathbb{B} : \ |\chi(0)| = \|\chi\|_{\mathbb{B}}\}$.

When $\overline{co}\ F(\cdot, \cdot)$ is almost continuous the OSL condition (defined just above) is successfully applied in [5, 6, 7].

In the next section we present a very short proof of the relaxation theorem for almost continuous FDI with OSL right-hand side.

Afterward we give an appropriate extension of the OSL condition in case of discontinuous FDI.

Further we will use the following assumption:

A. *F is bounded on the bounded sets and moreover, $F(\cdot, \alpha)$ it is measurable for every $\alpha \in \mathbb{B}$.*

2. One sided Lipschitz functional differential inclusion

In this section we will present a short proof of the so called relaxation theorem. We will follow (with modification) the proof given in [3] in case of ordinary differential inclusions.

Consider the following functional differential inclusion:

$$\dot{x}(t) \in \overline{ext}\ F(t, x_t),\ x_0 = \varphi. \tag{1}$$

Theorem 2.1. *(Relaxation theorem) Let $F(t,\cdot)$ be continuous and OSL with a constant $L \geq 0$. If **A** holds, then the solution set of (1) is dense in the solution set of FDI.*

Proof. Since $F(\cdot,\cdot)$ is almost continuous, one has that $\overline{ext}\ F(\cdot,\cdot)$ is almost LSC (see Lemma 2.3.7 of [13]). Let $x(\cdot)$ be a solution of FDI. Fix $\varepsilon > 0$ and consider the following multifunction:

$$G_\varepsilon(t,\alpha) = \begin{cases} \{\overline{ext}\ F(t,\alpha)\} & \text{if } \alpha - x_t \notin \mathbb{B}_0 \\ \overline{\{v \in ext\ F(t,\alpha) :\ \langle x(t) - \alpha(0), \dot{x}(t) - v\rangle} & \\ \overline{< L|x(t) - \alpha(0)|^2 + \varepsilon\}}, \alpha \neq x_t & \text{if } \alpha - x_t \in \mathbb{B}_0 \\ \{Proj_{\dot{x}(t)}(\overline{ext}\ F(t,\alpha))\} & \text{if } \alpha \equiv x_t. \end{cases}$$

It is easy to see that $G_\varepsilon(\cdot,\cdot)$ is almost LSC with non-empty compact values. Indeed $G_\varepsilon(\cdot,\cdot)$ has nonempty values since $\overline{ext}\ F(t,\cdot)$ is OSL. Furthermore, let $f \in G_\varepsilon(t,\alpha)$ and let $\overline{ext}\ F(\cdot,\cdot)$ be LSC at (t,α). If $x_t - \alpha \notin \mathbb{B}_0$ then G_ε is LSC at (t,α). If $\alpha \equiv x_t$ then G_ε is LSC at (t,α). Let $\alpha - x_t \in \mathbb{B}_0$ and let $\alpha_i \to \alpha$ and $t_i \to t$. We assume that $\dot{x}(\cdot)$ is continuous at t. We have to consider only the case $x_t - \alpha_i \in \mathbb{B}_0$. Let $f \in \overline{ext}\ F(t,\alpha)$ be such that $\langle x(t) - \alpha(0), \dot{x}(t) - f\rangle < L|x(t) - \alpha(0)|^2 + \varepsilon$. It is easy to see that there exists a sequence of $f_i \in \overline{ext}\ F(t_i,\alpha_i)$ such that $\langle x(t_i) - \alpha_i(0), \dot{x}(t_i) - f_i\rangle < L|x(t_i) - \alpha_i(0)|^2 + \varepsilon$ when i is large enough. The other cases can be considered analogously. Therefore $G_\varepsilon(\cdot,\cdot)$ is almost LSC and hence the differential inclusion:

$$\dot{y}(t) \in G_\varepsilon(t, y_t),\ y_0 = \varphi \tag{2}$$

has a solution $y(\cdot)$ (see [8]). Denote $r(t) = |x(t) - y(t)|^2$. Due to the definition of G_ε one has that $s(0) = 0$ and $\dot{s}(t) \leq Ls(t) + \varepsilon$ where $s(t) = \max\{r(t') :\ t' \in [t-\tau, t]\}$. Hence $s(t) \leq e^{|L|t}\varepsilon$. □

Notice that when $F(\cdot,\cdot)$ is discontinuous the OSL condition appears to be very weak. So we need the following stronger condition (which is, however, much weaker than the commonly used Lipschitz condition):

A1. (Strong one sided Lipschitz (SOSL) condition) There exists a constant $L \geq 0$ such that for every $\alpha, \beta \in \mathbb{B}$ (not necessarily $\alpha - \beta \in \mathbb{B}_0$)

$$\sigma(\alpha(0) - \beta(0), F(t,\alpha)) - \sigma(\alpha(0) - \beta(0), F(t,\beta)) \leq L\|\alpha - \beta\|^2_{\mathbb{B}}.$$

Denote $R(t,\alpha) = \bigcap_{\varepsilon>0} \overline{co}\, F(t, \alpha + \varepsilon U_{\mathbb{B}})$. We let R_{RD} to be the solution set of the relaxed FDI (RFDI):

$$\dot{x}(t) \in R(t, x_t),\ x_0 = \varphi,\ t \in J.$$

Theorem 2.2. *Let $F : I \times B \rightrightarrows E$ satisfy **A1**. If, moreover, F maps bounded sets into bounded, then there exists a constant C such that $D_H(R_{1\Delta}, R_{RD}) \le C\sqrt{h_\Delta}$, where $R_{1\Delta}$ is the solution set of the discrete inclusion:*

$$\dot{x}(t) \in F(t, x_i),\ x_i = x_{t_i},\ x_0 = \varphi,\ t \in [t_i, t_{i+1}],\ i = 0, 1, \dots, n-1 \tag{3}$$

and $h_\Delta = \max\limits_{1\le i\le n} \{t_{i+1} - t_i\}$ is the (maximal) step of the subdivision $\Delta := \{0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1\}$.

Remark 2.1. Obviously $R : J \times \mathbb{B} \rightrightarrows E$ is almost USC with (nonempty) convex compact values.

Proof. First we will see that there exist constants M and N such that every solution $x(\cdot)$ of

$$x(t) \in \overline{co\ R(t, x_t + U_{\mathbb{B}}) + U},\ x_0 = \varphi \tag{4}$$

satisfies $|x(t)| \le M$ and $|R(t, x_t + U_{\mathbb{B}}) + U| \le N$.

Notice first that the multifunction $G(t,\alpha) \equiv \overline{co\ R(t, \alpha + U_{\mathbb{B}}) + U}$ is almost USC and SOSL with a constant L. Moreover, it is bounded on the bounded sets.

Let $x(\cdot)$ be a solution of (4). Hence:

$$\langle \dot{x}(t), x(t)\rangle \le \sigma(x(t), G(t, x_t)) \le L\|x_t\|^2_{\mathbb{B}} + \sigma(x(t), G(t, 0)).$$

If $x_t \in \mathbb{B}$, then $\langle \dot{x}(t), x(t)\rangle \le L|x(t)|^2 + |x(t)||G(t,0)|$,

i.e. $\frac{d}{dt}|x(t)| \le L|x(t)| + |G(t,0)|,\ x(0) = \varphi.$

Thus there exists M such that $|x(t)| \le M$. Since G is bounded on the bounded sets the constant N also exists.

Hence if $Nh_\Delta < 1$ one has that every solution of the discrete FDI (3) satisfies $|x(t)| \le M$.

Using standard arguments one can show that $\sigma(\alpha(0) - \beta(0), R(t,\alpha)) - \sigma(\alpha(0) - \beta(0), F(t,\beta)) \le L\|\alpha - \beta\|^2_{\mathbb{B}}$.

Taking a subdivision Δ of J with $h_\Delta < \frac{\varepsilon}{N}$ one has that $\dot{x}(t) \in R(t, x_t + \varepsilon U_{\mathbb{B}})$. Let $y(\cdot) \in R_{RD}$. For $i = 0, 1, \dots, n-1$, we choose $f(t) \in F(t, x_i)$

on $[t_i, t_{i+1}]$, where $x_i = x_{t_i}$ and $x(t) = x_i(0) + \int_{t_i}^{t} f(s)\,ds$, such that $\langle y(t) - x_i(0), \dot{y}(t) - f(t)\rangle \le L\|y_t - x_i\|^2_{\mathbb{B}}$. One can show that $\frac{d}{dt}|x(t)-y(t)|^2 \le 2L\|x_t - y_t\|^2_{\mathbb{B}} + 4|L|Nh_\Delta(4M+Nh_\Delta)$. Consequently $\frac{d}{dt}|x(t)-y(t)|^2 \le 2L|x(t)-y(t)|^2 + 4|L|Nh_\Delta(4M+Nh_\Delta)$ when $y_t - x_t \in \mathbb{B}_0$. From Lemma 2 of[12] we know that $|x(t)-y(t)|^2 \le r(t)$, where $\dot{r}(t) = 2Lr(t) + 4|L|Nh_\Delta(4M+Nh_\Delta)$, $r(0) = |\varphi - \psi|$ if $x_0 = \varphi$ and $y_0 = \psi$. Thus there exists a constant C such that $|x(t)-y(t)| \le C\sqrt{h_\Delta}$.

Let $x(\cdot)$ be a solution of (3), corresponding to the subdivision Δ. Therefore $|x(t)-y(t)| \le C\sqrt{h_\Delta}$. Hence $D_H(R_{1D}, R_{RD}) \le C\sqrt{h_\Delta}$. □

For $A \subset \mathbb{B}$ we let:

$$\Big(I + hR_t\Big)(A) = \Big[\alpha \in \mathbb{B},\ \exists \varphi \in A:\ \alpha(s) = \varphi(0)$$
$$+ \int_{t-h}^{t+s} R(s,\varphi)\,ds,\ s \in [-h,0] \text{ and } \alpha(s) = \varphi(s+h),\ s \in [-\tau,-h]\Big],$$

while $\Big(I + hR_t\Big)(A)(0) = \overline{\{a \in E:\ \exists \alpha \in \Big(I + hR_t\Big)(A),\ a = \alpha(0)\}}$. Let $S_R(t)$ be the reachable set at the time t of the relaxed FDI.

Let $F(\cdot)$ be autonomous, i.e. it does not depend on t. Therefore $R(\cdot)$ is also autonomous. In this case $\Big(I + hR\Big)(A)(0) = \bigcup_{\varphi \in A} \{\varphi(0) + hR(\varphi)\}$. We denote it by $\Big(I + hR)(A)$. Define $Sol\Big[\Big(I + hR\Big)(A)\Big]$ to be the set of $\psi \in \mathbb{B}$ for which there exist $v \in A$ and $f \in R(A)$ such that

$$\psi(t) = \begin{cases} v(0) + (t+h)f,\ t \in [-h,0] \\ v(t+h),\ t \in [-\tau,-h] \end{cases}$$

Furthermore we define recurrently

$$\Big(I + hR\Big)^{n+1}(A) = \Big(I + hR\Big)\Big(Sol\Big[\Big(I + hR\Big)^{n}(A)\Big]\Big).$$

The following corollary extends the exponential formula of Wolenski (compare with [14]) to the case of SOSL functional differential inclusions.

Corollary 2.1. (Exponential formula) *Under the conditions of Theorem 2.2 if $F(\cdot)$ is autonomous, then for every $t \in [0,1]$ one has:*

$$S_R(t) = \lim_{n\to\infty} \Big(I + \frac{t}{n}R\Big)^{n}(\varphi) := e^{tR}\varphi.$$

P r o o f. For $i = 0, 1, \dots, n-1$, consider the discrete FDI:

$$x(t) = x_i(0) + (t - t_i)f,\ f \in R(x_i),\ x_i = x_{t_i},\ x_0 = \varphi,\ t_i = \frac{i}{n},\ t \in [t_i, t_{i+1}].$$

The reachable set of this inclusion coincides in the points t_i with the reachable set of

$$\dot{x}(t) \in R(x_i),\ x_0 = \varphi,\ x_i = x_{t_i},\ t \in [t_i, t_{i+1}],\ i = 0, 1, \dots, n-1.$$

It is easy to see that the Hausdorff distance between the solution sets of these inclusions is $O\Big(\frac{1}{n}\Big)$. Consequently

$$D_H\Big(S_R(t); \Big(I + \frac{t}{n}R\Big)^n(\varphi)\Big) \le C\sqrt{\frac{1}{n}}. \qquad \square$$

Now we will study the case of almost LSC $F(\cdot,\cdot)$. Namely assume:
A2. The multifunction $F(\cdot,\cdot)$ is almost LSC with nonempty compact values.

Theorem 2.3. *Let all the conditions of Theorem 2.2 hold. Under* ***A2*** *the solution set of (FDI) is nonempty and connected. Furthermore it is dense in the solution set of (RFDI).*

P r o o f. The existence of solution is proved in [8]. First we will prove the density. Let $x(\cdot)$ be a solution of (RFDI). Fix $\varepsilon > 0$ and consider the multifunction:

$$G_\varepsilon(t, \alpha) = \overline{\{v \in ext\ F(t, \alpha) :\ \langle x(t) - \alpha(0), \dot{x}(t) - v\rangle < L|x_t - \alpha|^2 + \varepsilon\}}$$

It is easy to see that $G_\varepsilon(\cdot,\cdot)$ is almost LSC with nonempty compact values. Therefore there exists a solution $y(\cdot)$ of the functional differential inclusion (2). Furthermore $\frac{d}{dt}|x(t) - y(t)|^2 \le 2L|x(t) - y(t)|^2 + 2\varepsilon$, when $x_t - y_t \in \mathbb{B}_0$. Consequently $|x(t) - y(t)|^2 \le 2e^{2Lt}\varepsilon$. The density is therefore proved.

The fact that the solution set of (FDI) is connected can be proved in a similar way as in case of ordinary differential inclusion (cf. [4]). $\square$

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