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On the Limit q-Bernstein Operator

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Let $B_n(f, q; x)$, $n = 1, 2, \ldots$ be q-Bernstein polynomials of a function $f \in C[0, 1]$. The polynomials $B_n(f, 1; x)$ are classical Bernstein polynomials. Convergence properties of q-Bernstein polynomials for $q \neq 1$ differ essentially from those of the classical ones. In the case $0 < q < 1$ any sequence $\{B_n(f, q; x)\}$ generates a positive linear operator on $C[0, 1]$, which is called the *limit q-Bernstein operator*. The paper deals with some properties of this operator.

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1. Introduction

Due to the importance of Bernstein polynomials, their various generalizations have been studied (see, e.g. Lorentz [3], Lupaş [4], Petrone [6], Videnskii [9]).

Generalized Bernstein polynomials based on the q-integers, or *q-Bernstein polynomials* were introduced by Phillips [7] in 1997. If $q = 1$, these polynomials coincide with the classical ones. q-Bernstein polynomials have been studied by G. M. Phillips et al., who obtained a great number of results related to these polynomials, see Phillips [8] and references therein. Convergence of q-Bernstein polynomials was investigated by A. Il'inskii and S. Ostrovska in [2], [5]. In the case $q \in (0, 1)$ a sequence of q-Bernstein polynomials generates a linear operator on $C[0, 1]$, which we call the *limit q-Bernstein operator.* In the present paper we study some properties of this operator.

2. Preliminaries

To formulate our results we recall the following definitions (cf. [8]) .

Let $q > 0$. For any $n = 0, 1, 2, \ldots$, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \ldots + q^{n-1} \ (n = 1, 2, \ldots), \ [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \dots [n]_q \ (n = 1, 2, \dots), \ [0]_q! := 1.$$

For integers $0 \le k \le n$ the *q-binomial,* or the Gaussian coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \ \ [n]_1! = n!, \ \ \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

We denote by $C[0,1]$ the space of all continuous complex-valued functions on [0,1] equipped with the uniform norm. The expression $g_n(x) \rightrightarrows g(x)$ means convergence in $C[0,1]$.

Definition. ([7]) Let $f : [0,1] \to \mathbf{C}$, $q > 0$. The *q-Bernstein polynomial* of f is

$$B_n(f, q; x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q; x), \quad n = 1, 2, \dots,$$

where

$$p_{nk}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1 - q^s x), \quad k = 0, 1, \dots, n; \quad n = 1, 2, \dots$$

(From here on an empty product is taken to be equal 1.)

Note that the polynomials $B_n(f, 1; x)$ are classical Bernstein polynomials.

Below we state some properties of q-Bernstein polynomials which will be used throughout the paper. Their proofs can be found in [8], Ch. 7 and [2].

It follows directly from the definition that q-Bernstein polynomials possess the *end-point interpolation* property, in other words:

(1) $B_n(f, q; 0) = f(0)$, $B_n(f, q; 1) = f(1)$ for all $q > 0$ and all $n = 1, 2, \dots$.

It was proved in [7] that q-Bernstein polynomials reproduce linear functions, that is:

(2) $$B_n(at + b, q; x) = ax + b \ \text{ for all } \ q > 0 \ \text{ and all } \ n = 1, 2 \dots$$

Taking $a = 0$, $b = 1$ in (2), we conclude that

(3) $$\sum_{k=0}^{n} p_{nk}(q; x) = 1 \ \text{ for all } \ q > 0 \ \text{ and all } \ n = 1, 2, \dots$$

To describe the behavior of the sequence $\{B_n(f,q;x)\}$ in the case $q\in(0,1)$ and $n\to\infty$, consider the entire functions $p_{\infty k}(q;x) := \lim_{n\to\infty} p_{nk}(q;x)$, that is

$$(4)\quad p_{\infty k}(q;x) = \frac{x^k}{(1-q)^k [k]_q!}\prod_{s=0}^{\infty}(1-q^s x) =: \frac{x^k}{(1-q)^k[k]_q!}\psi(x),\ k=0,1,\ldots$$

By Euler's Identity (cf. [1], Ch. 2, Cor. 2.2), we have:

$$(5)\qquad \sum_{k=0}^{\infty} p_{\infty k}(q;x) \ = \ 1 \ \text{ for all } \ x\in[0,1).$$

For $f\in C[0,1]$, we set:

$$(6)\qquad B_\infty(f,q;x) := \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)p_{\infty k}(q;x), & \text{for } x\in[0,1),\\ f(1), & \text{for } x=1 \end{cases}$$

The following statement is true.

Theorem. *([2]). For $q\in(0,1)$ and any $f\in C[0,1]$,*

$$(7)\qquad B_n(f,q;x) \rightrightarrows B_\infty(f,q;x) \ \ for \ \ x\in[0,1] \ \text{ as } \ n\to\infty.$$

The equality $B_\infty(f,q;x)=f(x)$ holds if and only if $f(x)=ax+b$.

Therefore, in the case $q\in(0,1)$ the sequence $\{B_n(f,q;x)\}$ is not an approximating sequence for f unless f is linear. This is in contrast to the case $q=1$, when the sequence $\{B_n(f,1;x)\}$ approximates f for any $f\in C[0,1]$.

Definition. Let $q\in(0,1)$. The linear operator on $C[0,1]$ given by

$$B_{\infty,q} : f \mapsto B_\infty(f,q;x)$$

is called the *limit q-Bernstein operator.*

The theorem above shows that this operator arises naturally when we consider the limit of a sequence of q-Bernstein polynomials $(0<q<1)$.

The limit q-Bernstein operator admits a probabilistic interpretation. Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, and $X:\Omega\longrightarrow\mathbf{C}$ be a random variable. We use the standard notation $\mathsf{E}X$ for the mathematical expectation of X. Consider a sequence of discrete random variables $Y_n(q;x)$ having the distributions

$$(8)\qquad \mathsf{P}\left\{Y_n(q;x)=\frac{[k]_q}{[n]_q}\right\} = p_{nk}(q;x), \quad k=0,1,\ldots n; n=1,2,\ldots$$

(We can consider $p_{nk}(q;x)$ as probabilities since they are non-negative on [0,1] and satisfy (3)). Evidently, $B_n(f,q;x) = \mathsf{E}f(Y_n(q;x))$. The limits as $n \to \infty$ of both the values of $Y_n(q;x)$ and the probabilities of these values exist. Indeed,

$$\lim_{n\to\infty} \frac{[k]_q}{[n]_q} = 1 - q^k, \quad \lim_{n\to\infty} p_{nk}(q;x) = p_{\infty k}(q;x), \; k = 0,1,\ldots.$$

Moreover, for $x \in [0,1)$, we have $p_{\infty k}(q,x) \geq 0, \; k = 0,1,\ldots$; and $\sum_{k=0}^{\infty} p_{\infty k}(q;x) = 1$ because of (5).

Therefore, we can consider the random variables $Y_\infty(q;x)$ having the following probability distributions:

$$\begin{aligned} \mathsf{P}\{Y_\infty(q;x) = 1 - q^k\} &= p_{\infty k}(x), \; k = 0,1,\ldots \text{ for } x \in [0,1), \\ \mathsf{P}\{Y_\infty(q;1) = 1\} &= 1. \end{aligned} \tag{9}$$

Clearly, $B_\infty(f,q;x) := \mathsf{E}f(Y_\infty(q;x))$. The statement (7) implies that $Y_n(q;x) \to Y_\infty(q;x)$ in distribution.

3. Properties of the limit q-Bernstein operator

In this paper we give a survey of results on the properties of $B_{\infty,q}$. It includes new results as well as those already available. We start with the following statement.

Theorem 1. *$B_{\infty,q}$ is a bounded shape-preserving positive linear operator on $C[0,1]$, which leaves invariant linear functions.*

P r o o f. It is clear from (5) that $\|B_{\infty q}\| = 1$. Since $p_{\infty k} \geq 0, k = 0,1,2,\ldots$ on [0,1], it follows that $B_{\infty,q}$ is a positive linear operator. Using (2) and (7), we get

$$B_\infty(at+b,q;x) = ax + b, \tag{10}$$

that is $B_{\infty,q}$ leaves invariant linear functions. The operator $B_{\infty,q}$ is shape-preserving due to the fact that $B_{n,q}(f) := B_n(f,q;x)$ are shape-preserving linear operators (the proof is given in [8], Ch.7, Theorems 7.5.8 and 7.5.9). ∎

Theorem 2. *For any $f \in C[0,1]$,*

$$B_\infty(f,q;x) \rightrightarrows f(x) \;\; for \;\; x \in [0,1] \;\; as \;\; q \to 1^-.$$

P r o o f. With Korovkin's Theorem and (10), it suffices to show that

$$B_\infty(t^2,q;x) \rightrightarrows x^2 \;\text{ for }\; x \in [0,1] \;\text{ as }\; q \to 1^-.$$

Direct calculations give: $B_\infty(t^2, q; x) = x^2 + (1-q)x(1-x)$ and the statement now follows. ∎

R e m a r k . Using the probabilistic interpretation of $B_{\infty,q}$ we may prove Theorem 2 by applying Chebyshev's Inequality.

The following theorem treats analytic properties of the function $B_\infty(f, q; x)$. It shows that these properties are affected by the smoothness of f at 1. Whenever $B_\infty(f, q; x)$ allows analytic continuation into a complex domain D, we denote the continued function by $B_\infty(f, q; z)$, $z \in \mathbf{C}$.

Theorem 3. *i) If f is a polynomial, then $B_\infty(f, q; x)$ is also a polynomial and $\deg B_\infty(f, q; x) = \deg f$.*

ii) For any $f \in C[0,1]$ the function $B_\infty(f, q; x)$ is continuous on [0,1] and admits analytic continuation into the unit circle $\{z : |z| < 1\}$.

iii) If f possesses m derivatives (from the left) at 1, then $B_\infty(f, q; x)$ can be continued analytically into the circle $\{z : |z| < q^{-m}\}$.

In particular, if f is infinitely differentiable (from the left) at 1, then $B_\infty(f, q; z)$ is entire.

iv) If f possesses m derivatives and $f^{(m)}(x)\,(m \geq 0)$ satisfies the Lipschitz condition at 1, that is

$$(11) \qquad |f^{(m)}(x) - f^{(m)}(1)| \leq M(1-x)^\alpha \;\; for \;\; some \;\; M > 0, \;\; 0 < \alpha \leq 1,$$

then $B_\infty(f, q; x)$ can be continued analytically into the circle $\{z : |z| < q^{-(m+\alpha)}\}$.

P r o o f. Statements *i) - ii)* are proved in [2]. It is worth mentioning that, in general, $B_\infty(f, q; x)$ may not be differentiable at 1. For example, let $f \in C[0,1], f(0) = f(1) = 0$, and $f(1-q^k) = 1/k, k = 1, 2, \ldots$ Plain calculations show that

$$\lim_{h \to 0^+} h^{-1} \left[B_\infty(f, q; 1) - B_\infty(f, q; 1-h)\right] = \infty.$$

iii) Suppose that f is m times differentiable from the left at 1. By Taylor's formula

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(1)}{j!}(x-1)^j + r_m(x) =: T_m(x) + r_m(x),$$

where $T_m(x)$ is a polynomial and the remainder r_m is estimated by

$$(12) \qquad |r_m(x)| \leq C_m(1-x)^m \;\; \text{for all} \;\; x \in [0,1].$$

Obviously, $B_\infty(f, q; x) = B_\infty(T_m, q; x) + B_\infty(r_m, q; x)$. Since, by *i)* $B_\infty(T_m, q; x)$ is a polynomial, it suffices to prove that $B_\infty(r_m, q; x)$ can be analytically continued into $\{z : |z| < q^{-m}\}$. Using (4) and (6), we get

$$(13) \qquad B_\infty(r_m, q; z) = \psi(z) \sum_{k=0}^{\infty} \frac{r_m(1-q^k)\, z^k}{(1-q)^k [k]_q!}, \;\; |z| < 1,$$

where $\psi(z)$ is an entire function. It can be readily seen that

$$\lim_{k\to\infty}(1-q)^k[k]_q! = \psi(q) \neq 0.$$

Besides, we get from (12) that

$$|r_m(1-q^k)| \leq C_m q^{mk}, k = 0, 1, \ldots$$

Hence, the series in (13) converges for all $\{z : |z| < q^{-m}\}$.

iv) To prove the statement, we replace (12) with

$$|r_m(x)| \leq C_m(1-x)^{m+\alpha} \quad \text{for all} \quad x \in [0,1]$$

and obtain convergence of the series in (13) for all $\{z : |z| < q^{-(m+\alpha)}\}$. ■

R e m a r k . If the Lipshitz condition (11) is replaced by the logarithmic one, i.e.,

$$|f^{(m)}(x) - f^{(m)}(1)| \leq \frac{M}{\ln^\beta (1/(1-x))} \quad \text{for some} \quad M > 0, \quad \beta > 1, \quad x \to 1,$$

then $B_\infty(f, q; x)$ possesses $[\beta - 1]^*$ continuous derivatives in the closed circle $\{z : |z| \leq q^{-m}\}$. Here $[x]^*$ denotes the greatest integer strictly less than x.

Theorem 4. *For any $f \in C[0,1]$ the function $B_\infty((1-t)^m f, q; z)$ can be analytically continued into $\{z : |z| < q^{-m}\}$ and the following identity holds*

$$B_\infty((1-t)^m f, q; z) = \left[\prod_{j=0}^{m-1}(1-q^j z)\right] B_\infty(f, q; q^m z), \quad |z| < q^{-m}. \tag{14}$$

In particular,

$$B_\infty((1-t)^m, q; z) = \prod_{j=0}^{m-1}(1-q^j z). \tag{15}$$

It is remarkable that the limit q-Bernstein operator maps the binomial $(1-x)^m$ into the q-binomial $(1-x)(1-qx)\ldots(1-q^{m-1}x)$.

P r o o f. Since the function $(1-t)^m f$ is m times differentiable from the left at 1, by Theorem 3 (*iii*) the function $B_\infty((1-t)^m f, q; x)$ can be analytically extended into $\{z : |z| < q^{-m}\}$. At the same time Theorem 3 (*ii*) guarantees that $B_\infty(f, q; q^m x)$ can be analytically extended into $\{z : |q^m z| < 1\} = \{z : |z| < q^{-m}\}$. Therefore, the functions in both sides of (14) are analytic in $\{z : |z| < q^{-m}\}$. By the Uniqueness Theorem for analytic functions, it suffices to prove (14) only for $z = x \in [0,1)$.

Using (4) we notice that

$$p_{\infty k}(q;qx)=\frac{q^k}{1-x}p_{\infty k}(q;x),\quad k=0,1,\ldots \text{ for } x\in[0,1).$$

Hence, for $x\in[0,1)$,

$$B_\infty(f,q;qx)=\sum_{k=0}^{\infty}f(1-q^k)p_{\infty k}(q;qx)=$$

$$\frac{1}{1-x}\sum_{k=0}^{\infty}q^kf(1-q^k)p_{\infty k}(q;x)=\frac{1}{1-x}B_\infty((1-t)f,q;x),$$

that is

$$B_\infty((1-t)f,q;x)=(1-x)B_\infty(f,q;qx). \tag{16}$$

Applying (16) m times, we get (14). ∎

R e m a r k . If f is k times differentiable on [0,1], then the identity (14) holds in $\{z:|z|<q^{-(m+k)}\}$. In particular, if f is infinitely differentiable on [0,1], then (14) is true for all $z\in\mathbf{C}$.

Theorem 5. *Let $f\in C[0,1]$ admit analytic continuation into $\{z:|z-1|\le 1+\varepsilon\}$. Then $B_\infty(f,q;x)$ is an entire function representable by:*

$$B_\infty(f,q;z)=\sum_{k=0}^{\infty}\frac{(-1)^kf^{(k)}(1)}{k!}\left[\prod_{j=0}^{k-1}(1-q^jz)\right]\quad \textit{for all } z\in\mathbf{C}. \tag{17}$$

P r o o f. The fact that $B_\infty(f,q;x)$ is an entire function follows from Theorem 3 (*iii*). On the other hand, the series in (17) converges uniformly on any compact set in $\mathbf{C}$ and, therefore, defines an entire function. Hence, by the Uniqueness Theorem it suffices to prove that (17) holds for $z=x\in[0,1)$. Let

$$f(x)=\sum_{j=0}^{n}\frac{(-1)^jf^{(j)}(1)}{j!}(1-x)^j+r_n(x),\quad x\in[0,1). \tag{18}$$

We apply $B_{\infty q}$ to both sides of (18). Because of (15) to prove the theorem we must show that

$$B_\infty(r_n,q;x)\rightrightarrows 0 \text{ for } x\in[0,1) \text{ as } n\to\infty.. \tag{19}$$

We have

$$B_\infty(r_n,q;x)=\sum_{k=0}^{\infty}r_n(1-q^k)p_{\infty k}(q;x)=\sum_{k=0}^{\infty}\left[\sum_{j=n+1}^{\infty}\frac{f^{(j)}(1)q^{kj}}{j!}\right]p_{\infty k}(q;x).$$

Using the Cauchy estimates for the derivatives $f^{(j)}(1)$, we get

$$\left|\sum_{j=n+1}^{\infty}\frac{f^{(j)}(1)q^{kj}}{j!}\right| \leq \sum_{j=n+1}^{\infty}\frac{M}{(1+\varepsilon)^j}q^{kj} = \frac{M}{1-\frac{q^k}{1+\varepsilon}}\left(\frac{q^k}{1+\varepsilon}\right)^{n+1} \leq \frac{M_1}{(1+\varepsilon)^{n+1}}.$$

Finally, by applying (5), we get (19). ∎

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