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On a Conjecture of Choban, Gutev and Nedev [1]

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The present paper collects the most important results concerning a Conjecture of Choban, Gutev and Nedev

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In [1], generalizing a result of Nedev [3](see the Remark 1 below), Choban and Nedev prove the following

Theorem 1 *[Choban, Nedev] Every l.s.c. closed-and-convex valued mapping $\Phi : X \to 2^Y$, where X is a* **GO** *space (a subspace of a linearly ordered space) and Y is a reflexive Banach space, has a single valued continuous selection.* theorem

In connection with this result they state the following

Conjecture 1 *(Choban, Gutev, Nedev) For every T_1-space X the following (a) and (b) are equivalent:*

(a) X is strongly normal (i.e. collectionwise normal and countably paracompact),

(b) Every l.s.c. closed-and-convex valued mapping $\Phi : X \to 2^Y$, where Y is a Hilbert space, has a single valued continuous selection;

The implication $(b) \to (a)$ in the above Conjecture is true and relatively easy to establish, while the implication $(a) \to (b)$ is still an open question. If proved, Conjecture 1 would present a characterization of *strong normality* by means of selections, completely analogous to the famous characterization of *paracompactness* due to E. Michael [2,Theorem 3.2"].

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Theorem 2 *(E. Michael, [2]) For a T_1-space X the following conditions are equivalent:*

(a) X is paracompact,

(b)Every l.s.c. closed-and-convex valued mapping $\Phi : X \to 2^Y$, where Y is a Banach space, has a single-valued continuous selection;

Strengthening the hypothesis in (a), we obtained some results (already published) confirming Conjecture 1:

Proposition 1 *Every l.s.c. closed - and - convex valued mapping $\Phi : X \to 2^Y$ has a single valued continuous selection in the following cases:*

(1) X is strongly normal and pseudo - paracompact, and Y is a reflexive Banach space [6],

(2) X is a Σ-product of metric spaces, and Y is a Hilbert space [5],

(3) X is a strongly normal space with no uncountable discrete subset, and Y is a Hilbert space [4];

A space X is called *pseudo - paracompact*if its topological completion is paracompact.

The *Σ-product* of a family of topological spaces $\{X_s\}_{s\in S}$ with the base point $x = \{x_s\} \in \prod_{s\in S} X_s$ is the subspace

$$\Sigma(x) = \{y = \{y_s\} \ : \ |\{s \in S \ : \ x_s \neq y_s\}| \leq \aleph_0\}$$

of the Tykhonov product $\prod_{s\in S} X_s$

Recently Nedev factorized the Conjecture 1 into two questions:

Question 1 *Is there a cardinal number τ such that every l.s.c. closed-and-convex valued $\Phi : X \to 2^Y$ with Y a Hilbert space and X a collectionwise normal and τ-paracompact space, has a single valued continuous selection?*

Question 2 *Is $\aleph_0$ (equivalently- is every infinite cardinal) among the cardinal numbers described in Question 1?*

Theorem 3 that follows answers Question 1 (in positive) by showing that $\mathfrak{c}$ (the cardinality of the continuum) is among the cardinal numbers described in Question 1. This obviously implies that any cardinal number that is larger than $\mathfrak{c}$ is also among the cardinals described in Question 1.

Theorem 3 *Every l.s.c. closed-and-convex valued mapping $\Phi : X \to 2^Y$, where X is collectionwise normal and $\mathfrak{c}$-paracompact space and Y is a reflexive Banach space, has a single-valued continuous selection.*

Remark 1 As it was shown by Nedev [3], every closed-and-convex valued mapping $\Phi : \omega_1 \to 2^Y$, where ω_1 is the well ordered space of all countable ordinals endowed with the standard order topology and Y is a reflexive Banach space, has a single valued continuous selection. Thus, the collectionwise normality and $\mathfrak{c}$-paracompactness in Theorem 3 is a sufficient but not a necessary condition for the existence of such a selection (ω_1 is strongly normal but not $\mathfrak{c}$-paracompact, even not $\aleph_1$-paracompact).

May be the most interesting situation arose when we met the following characterization of strong normality due to J.C. Smith:

Theorem 4 *(J.C. Smith, [8]) A space X is strongly normal iff every weak θ-cover of X has an open locally finite refinement.*

It turns out that Conjecture 1 is a consequence of Theorem 4. Unfortunately, we found an incompleteness in Smith's proof of Theorem 4, and we still do not know how to remove it.

We were able only to obtain the following weaker version of Theorem 4:

Theorem 5 *[7] Let X be a hereditarily collectionwise normal and countably paracompact space. Then every weak θ-cover of X has an open locally finite refinement.*

and the corresponding consequence:

Theorem 6 *[7] Let X be a hereditarily collectionwise normal and countably paracompact space, Y be a reflexive Banach space, and $\Phi : X \to 2^Y$ be an l.s.c. closed-and-convex valued mapping. Then Φ has a single-valued continuous selection.*

Since every **GO** space is hereditarily collectionwise normal, countably paracompact and pseudo - paracompact, we can observe that both Theorem 6 and Proposition 1 (in case (1)) generalize Theorem 1 but in different directions.

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