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The Leibniz Programme: Calculation in Lieu of Disputation

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1. Historical introduction

Leibniz's basic idea of the arithmetization of human reasoning was related to syllogistic. His goal was to establish a correspondence between syllogistic terms (or notions) and suitable integers (their *characteristic numbers*), so that the logical truth of a proposition would turn into an arithmetical truth of a calculation. This idea had two realizations described by Leibniz. The first one used *single* integers and their divisibility but unfortunately was unsuccessful. The second one was successful but more complicated and less intuitive: it used *pairs* of co-prime numbers (see [Łuk 57, § 34]). Leibniz's texts can be found in Couturat's old edition [Cou 03] as well in the new academic edition of his philosophical writings [Lei 99].

Here I will justify the viability of the earlier, less complicated realization (with appropriate modifications, of course). Something more, Leibniz's initial sketch will be extended so to envelop the syllogistic enriched with *term negation* and *term composition* (that is the whole Boolean syllogistic), the *monadic predicate calculus* (i.e., the logic of properties), and the *monadic predicate calculus with equality* as well. Various syllogistic systems have clear algebraic representation in the terms of *partially-ordered structures*, *semi-lattices, lattices*, and *Boolean algebras* (see my report [Sot 99c]).

The full proof of the theorems concerning syllogistic together with a detailed historical and bibliographical exposition of Leibniz's logical ideas can be found in [Sot 99a]. The arithmetization of the pure monadic predicate calculus was announced in [Sot 99b]; the full proof is published here. The arithmetization

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of the monadic predicate calculus with equality was exposed in [Sot 01]. Almost all results mentioned in this report are accessible from my Web-site.

2. Arithmetizations of the traditional syllogistic

I will treat the Aristotelian syllogistic in the style that became canonical after Łukasiewicz's celebrated book [Łuk 57]. For this purpose the language of the classical propositional calculus is extended by *term variables* (for short, *terms*) $t_1, t_2, \ldots$ together with two binary *term relations*: $\mathcal{A}$ and $\mathcal{I}$. Syllogistic *atoms* are all formulae of the kind $s\mathcal{A}p$ or $s\mathcal{I}p$ with s and p being terms. A *syllogism* is any propositional formula with all propositional letters replaced by syllogistic atoms.

The standard and the most intuitive semantics of the Aristotelian syllogistic is that in the theory of sets: if S and P are arbitrary *non-empty* sets, $s\mathcal{A}p$ is translated into $S \subseteq P$, $s\mathcal{I}p$ into $S \cap P \neq \emptyset$, and the formal propositional connectives are replaced with the informal ones. Thus any syllogism is translated into a sentence about non-empty sets. If this sentence is true, i.e., if the expression so obtained is a set-theoretical tautology, the syllogism is said to be *true.* It is *true in a given* (*non-empty*) *set* U when any replacement of its terms with (non-empty) subsets of U gives a true sentence. I call this semantics *Scholastic* according to Leibniz's use of this word. It can be shortly characterized by the pair $(\subseteq, \cap \neq \emptyset)$.

Another semantics in the theory of sets is also possible; it will be named *Leibnizian* being (partially) accepted by him. When a non-empty set U is given, term variables are evaluated by subsets of U *different from* U. If S and P are such sets, $s\mathcal{A}p$ is interpreted as $S \supseteq P$, $s\mathcal{I}p$ as $S \cup P \neq U$, and the formal propositional connectives are replaced by informal ones. A syllogism is said to be *true in* U when the sentence obtained after any replacement of all term variables with subsets of U (different from U) is true. The syllogism is *true* when it is true in any set U. This semantics is characterized by $(\supseteq, \cup \neq U)$. Obviously, both semantics are dual.

On the base of both set-theoretical intuitions, two translations of syllogistic are obtained:

Scholastic arithmetical interpretation. Let $a_1, a_2, \ldots$ denote arbitrary integers *greater than* 1. Given a syllogism, replace $t_i\mathcal{A}t_j$ with $a_i|a_j$ ("a_i is a divisor of a_j"), $t_i\mathcal{I}t_j$ with a new relation a_iGa_j ("a_i and a_j have a common divisor greater than 1", or: g.c.d. $(a_i, a_j) > 1$), and the formal propositional connectives with the informal ones. In short, this interpretation is characterized by $(|, \text{g.c.d.} > 1)$. Call the syllogism *arithmetically true* (in the *Scholastic sense*) if the sentence so obtained is an arithmetical truth.

Theorem 1 (Adequacy of the Scholastic arithmetical interpretation): *A syllogism is true iff it is arithmetically true in the Scholastic sense.*

If the empty set is admitted to evaluate terms in the Scholastic semantics, the arithmetical interpretation may be modified: 1 has to be added to the list of divisors being the number corresponding to the empty terms.

Leibnizian arithmetical interpretation. Let u (the *Universe number*) be an arbitrary integer greater than 1, and let $a_1, a_2, \ldots$ be arbitrary its proper divisors, i.e., $a_i < u$ for any i (however, $a_i = 1$ is permitted). Replace $t_i \mathcal{A} t_j$ by a relation a_i / a_j ("a_i is divisible by a_j"), and $t_i \mathcal{I} t_j$ by a relation $a_i L a_j$ ("the least common multiple of a_i and a_j is less than u", or: "there is a *prime* divisor of u dividing neither a_i nor a_j"). In short, the Leibnizian arithmetical interpretation is characterized by ($/$, l.c.m.$< u$). Finally, formal propositional connectives are replaced with their informal analogues. The syllogism is said to be *arithmetically true* (in the *Leibnizian sense*) *with respect to u* if the sentence so obtained is an arithmetical truth. The syllogism is *arithmetically true* (in the *Leibnizian sense*) if it is arithmetically true in the same sense with respect to any $u > 1$.

Theorem 2 (Adequacy of the Leibnizian arithmetical interpretation): *A syllogism is true iff it is arithmetically true in the Leibnizian sense.*

3. Arithmetizations of syllogistic with negative terms

Expand the language of syllogistic by adding an operation of *term negation* $-$; then, if t is a term, $-t$ ("non-t") is a term, too. The definition of atoms is modified by permitting s and p to be arbitrary terms in $s\mathcal{A}p$ and in $s\mathcal{I}p$ as well. In both set-theoretical semantics, a *universal set* U is introduced. According to the tradition, terms are evaluated by subsets of U different from $\emptyset$ and U (in such a case, U obviously cannot be an one-element set). If a term t is evaluated by a set T, the value of $-t$ is the *complement* of T to U. The rest of the definitions of a true syllogism remains the same. In both arithmetical interpretations, a *Universe number* $u > 1$ *without multiple factors* is introduced together with the following rules: 1) all evaluating integers are divisors of u different from 1 and u (therefore, u cannot be prime); 2) if term t is evaluated by an integer a, then the term $-t$ is evaluated by $\frac{u}{a}$.

Theorem 3 (Adequacy of both Scholastic and Leibnizian arithmetical interpretations of syllogistic with term negation): *A syllogism (possibly with negative terms) is true iff it is arithmetically true in the Scholastic as well as in the Leibnizian sense.*

4. Arithmetizations of syllogistic with term composition

In this section, neither empty not universal sets will been rejected. The treatment may be made independent of the presence of term negation. However, if negation does occur together with a composition (it does not matter whether it will be treated as an intersection or as a union), all Boolean term operations will be defined. That is why it will be better to consider the full Boolean algebra straight away.

The composition will be noted by $\circ$. The class of terms now is the smallest class including term variables, and closed under *negation* and *composition*. Given a Universe $U \neq \emptyset$, an evaluation of a term t in U is a set T obtained after replacing all term variables in t with arbitrary subsets of U as well as term operations with their corresponding set-theoretical operations. Namely, in the Scholastic semantics $\circ$ is interpreted as an *intersection*, and in the Leibnizian semantics it is a *union*. Having terms evaluated, the translation of a syllogism into a set-theoretical sentence remains the same as in Section 2.

Further, in both arithmetical interpretations, term variables will be evaluated by arbitrary divisors of a Universe number $u > 1$ *without multiple factors*. The evaluation of the negation remains as it was defined in the previous Section. If terms s_1 and s_2 are evaluated by integers a and b, the composition $s_1 \circ s_2$ will be modelled by g.c.d. (a, b) in the Scholastic arithmetical interpretation, and by l.c.m. (a, b) in the Leibnizian one. Note that in the second case, if a and b have no common divisor, $s_1 \circ s_2$ is represented by their product ab.

Theorem 4 (Adequacy of both Scholastic and Leibnizian arithmetical interpretations of syllogistic with all Boolean term operations): *A syllogism (possibly containing arbitrary Boolean term operation) is true iff it is arithmetically true in the Scholastic as well as in the Leibnizian sense.*

5. Arithmetization of the pure monadic predicate calculus

The language of the monadic calculus contains *individual variables* x, y, z, ..., one-place *predicate symbols* $P_1, P_2, ...$, *quantifiers* $\forall$ and $\exists$, and the usual propositional connectives with brackets. A *monadic proposition* is a formula without free variables. For such formulae we adopt Theorem 25.4 from [BJ 89]:

Lemma 1 *Any monadic proposition is equivalent to a monadic proposition with the same predicate symbols and one variable only.*

Let the sole variable be x. In addition, we may suppose it is not bound "twice" anywhere. So, no formula is a Boolean combination of two subformulae

$A(x)$ and B, one of them containing a *free* x and the other containing x *bound*; in $(Qx)A(x)$, where Q is a quantifier, a free x does occur in A. In other words, the formula under consideration either is of the kind $Q(x)A(x)$ where $A(x)$ does not contains quantifiers (and therefore is a Boolean combination of predicates), or is a Boolean combination of formulae of the kind just mentioned.

To build up an arithmetical model for monadic propositions, let an arbitrary integer $u > 1$ *without multiple factors* be taken, and let its divisor d_i be associated with the predicate $P_i(x)$. Further, following the construction of the formula, a *divisor* of u will be associated with any subformula containing a free x, and a *statement* about divisors will be associated with the subformula when it does not contain a free x. Namely, if a and b are associated with $A(x)$ and $B(x)$, then g.c.d. (a, b) is associated with $A(x)\&B(x)$, $\frac{u}{a}$ with $\neg A(x)$, and so on for other Boolean connectives; the statements $a = u$ and $a > 1$ are associated with $(\forall x)A(x)$ and $(\exists x)A(x)$, respectively; if statements p and q are associated with subformulae A and B, then "p and q" and "not p" will be associated with $A\&B$ and $\neg A$, respectively. Finally, a certain statement comparing divisors of u with u and 1 will model the initial monadic proposition. If this statement is an arithmetical truth for an arbitrary integer u, the proposition is called *arithmetically true*. Using that any predicate tautology is equivalent to a closed formula, we obtain the main

Theorem 5 *Any formula of the pure monadic predicate calculus is a tautology iff its corresponding monadic proposition is arithmetically true.*

Proof. For the proof the well known fact that the monadic predicate calculus is decidable is used. I. e., if given formula with n predicate letters is not valid, it is rejected in a domain D containing $N \leq 2^n$ elements. Suppose the elements of D are $e_1, ..., e_N$, the predicate P_i is interpreted by a subset D_i of D, and the rejecting evaluation attaches an element e of D to the sole individual variable x. As it was shown, the formula under consideration may be supposed to be a Boolean combination of subformulae of the kind $(Qx)A(x)$ where Q denotes a quantifier. Let A^* be the *sentence* obtained after replacing in $A(x)$ each predicate $P_i(x)$ by the expression $e \in D_i$ and each formal connective by its informal analogue. $A(x)$ is true under given evaluation in given interpretation *iff* A^* is true. Further on, let A^{**} denotes the *set* obtained after replacing the predicate $P_i(x)$ by D_i and the propositional connectives by their set-theoretical analogues (i. e., $\wedge$ by $\cap$, $\neg$ by complement, etc.). Then A^* is true *iff* $x \in A^{**}$ is true. Hence $(\forall x)A(x)$ will be true in given interpretation *iff* $A^{**} = D$, and $(\exists x)A(x)$ will be true *iff* $A^{**} \neq \emptyset$.

Now it is easy to show that the interpretations of a monadic predicate formula in sets and its arithmetical interpretations are isomorphic. Given an interpretation in a set D, take N *different prime* numbers $a_1, ..., a_N$ and let u be their product. If the subset D_i is not empty, denote by d_i the divisor of u obtained after multiplying those prime numbers which indexes coincide with the indexes of the elements of D_i; if D_i is empty, $d_i = 1$. Conversely, given an arithmetical model with a Universe u without multiple factors, let D denote the set of all prime multipliers of u, and let D_i be the set of the prime multipliers of its arbitrary multiplier d_i; if d_i is 1, $D_i = \emptyset$. Then for any sub-formula $A(x)$, A^{**} coincides with the divisor of u which is obtained as an arithmetical value of $A(x)$. It is obviously that that divisor will be equal to u (and respectively, $(\forall x)A(x)$ will be arithmetically true) *iff* $A^{**} = D$, i. e., *iff* $(\forall x)A(x)$ is true independently of the value of the variable x. The case of $(\exists x)A(x)$ is analogous. Therefore, any sub-formula of the kind $(Qx)A(x)$ is arithmetically true exactly when it is true in the corresponding interpretation in sets.

6. Arithmetization of the monadic predicate calculus with equality

Now, the language of the pure monadic predicate is extended by the only two-place predicate =. Formulas are defined in the usual way. For this language, *models* in non-empty domains and *evaluations* of individual variables in them are introduced in the standard manner [Kle 67, § 29].

For the *arithmetical* models, let $u > 1$ be an integer *without multiple factors*. Any predicate P_i is interpreted by arbitrary divisor of u (possibly 1 or u) denoted with $d(P_i)$, and any individual variable x_i is evaluated by $d(x_i)$, a *prime* divisor of u. Following the construction of a formula F, its *arithmetical statement* $\mathcal{AR}[F]$ *corresponding to given evaluation* will be obtained. Namely, for atomic formulas $\mathcal{AR}[P_i(x_j)]$ is "$d(x_j)$ divides $d(P_i)$" and $\mathcal{AR}[x_i = x_j]$ is "$d(x_i) = d(x_j)$"; for a subformula G, $\mathcal{AR}[(\forall x)G]$ is "for any prime divisor d, $\mathcal{AR}^x_d[G]$" where $\mathcal{AR}^x_d$ differs from $\mathcal{AR}$ only attaching d to x; finally, all propositional connectives are replaced with their non-formal analogues. If $\mathcal{AR}[F]$ is a true arithmetical sentence for any u under arbitrary evaluation, F is named *arithmetically true*. This semantics is relevant:

Theorem 6 *Any formula of the monadic predicate calculus with equality is a tautology iff it is arithmetically true.*

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