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Limit Theorem for High Level A-Upcrossings by χ-Process

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1. Introduction. Result

Properties of high level intersection sets by trajectories of Gaussian random processes on infinitely increasing time horizon are well elaborated, see [2], [3] and references therein. Many important results in this direction have been obtained for Gaussian fields, [3]. In contrast, there are only few results about limit behavior of the number of large excursions of Gaussian vector processes. First Poisson limit theorem for a-exit points over level u, where Gaussian vector process of arbitrary dimension was investigated, was established in [4]. The present paper deals with A-upcrossing (A-exit) points over high level u. We use similar technic as in [4]. We consider the stationary random process

$$\chi(t) = \left(X_1^2(t) + X_2^2(t) + ... + X_n^2(t)\right)^{1/2} = ||\mathbf{X}(t)||, \quad t \in \mathbb{R},$$

where $\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t))$ is a Gaussian vector process which components are independent copies of a Gaussian stationary process $X(t)$ with mean zero and covariance function $r(t)$. We assume that

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha) \text{ as } t \to 0, \text{ for some } 0 < \alpha \leq 2, \tag{1}$$

and

$$r(t)\log t \to 0 \text{ as } t \to \infty. \tag{2}$$

From (2) it follows that

$$\text{for any } \delta > 0, \sup_{|t|>\delta} |r(t)| < 1. \tag{3}$$

The aim of this paper is to prove a Poisson limit theorem for A-points of upcrossings of a high level by the trajectories of the process $\chi(t)$.

Definition 1. *A pair (A, ρ_A), $A \subset \mathbb{R}$, where A is a bounded Borel set and $\rho_A > 0$ is called a trap if*

1. *Relations $t \notin A$, $0 \notin A + t$ imply $|t| > \rho_A$;*
2. *One can find a point t in any non-empty closed bounded set $B \subset \mathbb{R}$ such that $(A + t) \cap B = \{t\}$.*

Definition 2. *Suppose a set $S \subset \mathbb{R}$ and a trap (A, ρ_A) are given. A point $t \in S$ is called an A-point of the set S if $(A + t) \cap S = \{t\}$.*

Definition 3. *Let a trap (A, ρ_A) be given. A point t is called an A-upcrossing of the level u by the process $\chi(t)$, $t \in \mathbb{R}$, if it is an A-point of the set $\{t : \chi(t) \geqslant u\}$.*

For simplicity we will use the following notations for maximum distributions,

$$P_X(u, W) = \mathbf{P}\left(\max_{t \in W} X(\mathbf{t}) \le u\right) \text{ and } \overline{P}_X(u, W) = 1 - P_X(u, W),$$

where $X(\mathbf{t})$ is a random process or field. Denote

$$\mu(u) = \frac{2^{(3-n)/2}\sqrt{\pi} H_\alpha}{\Gamma(n/2)} u^{2/\alpha + n - 1} \Psi(u),$$

where Γ is the gamma function, $\Psi(u) = \frac{1}{\sqrt{2\pi}} u^{-1} e^{-u^2/2}$ and H_α is the Pickands' constant. It is proven in [3], Corollary 7.3, that for any T,

$$\overline{P}_\chi(u, [0, T]) = T\mu(u)(1 + o(1)) \text{ as } u \to \infty. \tag{4}$$

Let us introduce a transformation of the space $\mathbb{R}$, $h_u t := \mu(u)^{-1} t$, $t \in \mathbb{R}$, $u > 0$. By $\mathcal{B}$ denote the σ-algebra of Borel sets on $\mathbb{R}$. Let $\eta_{A,u}(B)$, $B \in \mathcal{B}$, be the point process of A-upcrossings of the level u by the process $\chi(t)$. Consider the normalized point process

$$\Phi_u(B) := \eta_{A,u}(h_u B).$$

Let $\Phi(\cdot)$ be the standard Poisson point process on $\mathcal{B}$, that is, stationary with intensity one. Our main result is

Theorem 1. *Let assumptions (1,2) be fulfilled for the stationary random process $\chi(t)$. For any trap (A, ρ_A) the random point process $\Phi_u(B)$, $B \in \mathcal{B}$, converges weakly as $u \to \infty$ to the standard Poisson point process $\Phi(B)$, $B \in \mathcal{B}$.*

Let $\mathcal{L}$ be a sub-ring of $\mathcal{B}$, generated by semi-open intervals $[t, s)$. Let an infinitely divisible point process $\Phi(B)$ on $\mathcal{B}$ be given such that $\Phi(\partial L) = 0$ with probability one for all $L \in \mathcal{L}$. By Kallenberg theorem (Theorem 4.7 [1]), if for any $L \in \mathcal{L}$,

$$(5) \qquad \lim_{u\to\infty} \mathbf{P}(\Phi_u(L) = 0) = \mathbf{P}(\Phi(L) = 0) \text{ and } \limsup_{u\to\infty} \mathbf{E}\Phi_u(L) \leq \mathbf{E}\Phi(L),$$

then the weak convergence

$$\Phi_u(B) \Rightarrow \Phi(B)$$

takes place. We prove in section 2 the relations of (5) for the Poisson point process $\Phi(B)$.

2. Proofs

Lemma 1. *For the intensity $\mu_A(u)$ of the random point process $\eta_{A,u}(B)$ we have, $\lim_{u\to\infty} \mu_A(u)/\mu(u) = 1$.*

Proof. It is sufficient to evaluate the asymptotic behavior of the probability $\mathbf{P}(\eta_{A,u}(I) > 0)$ as $u \to \infty$, where $I = [0, 1]$. Note that

$$\{\eta_{A,u}(I) > 0\} \subset \left\{ \max_{t\in I} \chi(t) > u \right\}.$$

From the other hand,

$$(6) \qquad \{\eta_{A,u}(I) > 0\} \supset \left\{ \max_{t\in((I\oplus A)\setminus I)} \chi(t) \leqslant u, \max_{t\in I} \chi(t) > u \right\},$$

where $\oplus$ is the Minkovsky sum of sets, that is $A \oplus B = \{t + s : \ t \in A, s \in B\}$. Further,

$$\mathbf{P}\left\{ \max_{t\in A} \chi(t) \leqslant u, \max_{t\in I} \chi(t) > u \right\}$$
$$= \bar{P}_\chi(u, I) - \left[\bar{P}_\chi(u, I) + \bar{P}_\chi(u, I \oplus A) - \bar{P}_\chi(u, I \oplus A)\setminus I)\right].$$

The expression in square brackets is infinitely smaller then $\bar{P}_\chi(u, I)$ so Lemma follows. ∎

Lemma 2. *For any* $L \in \mathcal{L}$, $\mathbf{P}\left(\Phi_u(L) = 0\right) = P_\chi(u, h_u L) + o(1)$ *as* $u \to \infty$.

Proof. Denote $L_u = h_u L$, $L \in \mathcal{L}$. We have, $P_\chi(u, L_u) \le \mathbf{P}\left(\Phi_u(L) = 0\right)$. Further, similarly to (6) we have,

$$\mathbf{P}\left(\Phi_u(L) = 0\right) \le P_\chi(u, L_u) + \mathbf{P}\left(\max_{t \in L_u} \chi(t) > u, \max_{(L_u \oplus A)\setminus L_u} \chi(t) > u\right).$$

Since L consists of a finite number of intervals,

$$\mathbf{P}\left(\max_{L_u} \chi(t) > u, \max_{(L_u \oplus A)\setminus L_u} \chi(t) > u\right) \le \mathbf{P}\left(\max_{(L_u \oplus A)\setminus L_u} \chi(t) > u\right) = o(1).$$

as $u \to \infty$. The Lemma is proven. ∎

Introduce a Gaussian random field $Y(t, \mathbf{v}) = (\mathbf{X}(t), \mathbf{v}) = \sum_{j=1}^{n} X_j(t) v_j$, where

$$t \in \mathbb{R}, \mathbf{v} = (v_1, ..., v_n) \in S_{n-1} := \{(x_1, x_2, ..., x_n) : x_1^2 + x_2^2 + ... + x_n^2 = 1\}.$$

By duality, for any closed $T \subset \mathbb{R}$,

$$\max_{t \in T} \chi(t) = \max_{(t,\mathbf{v}) \in T \times S_{n-1}} Y(t, \mathbf{v}),$$

in particular, $\chi(t) = \max_{\mathbf{v} \in S_{n-1}} Y(t, \mathbf{v})$. By Bounyakovsky, it is easily to see that the last maximum is attained at only point of the sphere S_{n-1}, which corresponds a unit vector directed as $\mathbf{X}(t)$. Random field $Y(t, \mathbf{v})$, $(t, \mathbf{v}) \in \mathbb{R} \times S_{n-1}$ is homogeneous according the group of rotations on the sphere.

Lemma 3. *For every* $L \in \mathcal{L}$ *and any* $\epsilon > 0$ *one can find such* $b > 0$, $u_0 > 0$, $K = K(\epsilon) > 0$ *and a grid* $\mathcal{R}_{b,u,\epsilon}$ *on the cylinder* $L_u \times S_{n-1}$, *that for all* $u \ge u_0$,

$$P_Y(u, (L_u \times S_{n-1}) \cap \mathcal{R}_b) - P_Y(u, (L_u \times S_{n-1})) \le K\epsilon.$$

Proof. We show that the assertion holds true for $L = [0, T]$. Since in general L consists of finite number of intervals, the proof for general L will obviously follow. First we partition the sphere S_{n-1} onto $N(\epsilon)$ parts $A_1, ..., A_{N(\epsilon)}$ in the following way. Consider polar coordinates on the sphere S_{n-1},

$$(x_1, x_2, ..., x_n) = S(\varphi_1, \varphi_2, ...\varphi_{n-1}),$$

$\varphi_1, \varphi_2, ...\varphi_{n-2} \in [0, \pi)$, $\varphi_{n-1} \in [0, 2\pi)$ and divide the interval $[0, \pi]$ on intervals of length ϵ (or less for the last interval), do the same for the interval $[0, 2\pi]$. This partition of the parallelepiped $[0, \pi]^{n-2} \times [0, 2\pi]$ generates the partition $A_1, ..., A_{N(\epsilon)}$ of the sphere. Now we construct the grid $\mathcal{R}_{b,u,\epsilon}$. Choose in any A_j an inner point and consider the tangent plane to the cylinder $[0, T] \times S_{n-1}$ at the point. Introduce in the tangent plane rectangular coordinates, with origin at the tangent point, the first coordinate holds be t, so that the plane becomes $\mathbb{R}^n$, and consider the grids

$$\mathcal{R}^j(b) := \mathcal{R}^{j,P}_{b,u,\epsilon} := (bk_1u^{-2/\alpha}, bk_2u^{-1}, ..., bk_nu^{-1}), j = 1, 2, ..., N(\epsilon),$$

where $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$. Suppose that ϵ is so small that the orthogonal projection of all A_j onto corresponding tangent planes are one-to-one. Denote by $\mathcal{R}^j_{b,u,\epsilon}$, the prototype of $\mathcal{R}^{j,P}_{b,u,\epsilon}(b)$ under this projection. We show that the grid

$$\mathcal{R}_b := \mathcal{R}_{b,u,\epsilon} := \bigcup_{j=1}^{N(\epsilon)} \mathcal{R}^j_{b,u,\epsilon},$$

with appropriate choice of its parameters, satisfies the assertion of the lemma.

We have,

$$\begin{aligned} &P_Y(u, (L_u \times S_{n-1}) \cap \mathcal{R}_b) - P_Y(u, L_u \times S_{n-1}) \\ &= \mathbf{P}\left(\bigcap_{j=1}^{N(\epsilon)} \left(\max_{(L_u \times A_j) \cap \mathcal{R}_b} Y(t, \mathbf{v}) \le u\right) \cap \bigcup_{j=1}^{N(\epsilon)} \left(\max_{L_u \times A_j} Y(t, \mathbf{v}) > u\right)\right) \\ &\le \sum_{j=1}^{N(\epsilon)} \mathbf{P}\left(\max_{(L_u \times A_j) \cap \mathcal{R}_b} Y(t, \mathbf{v}) \le u, \max_{L_u \times A_j} Y(t, \mathbf{v}) > u\right). \end{aligned} \tag{7}$$

Denote by A_j^P the projection of part A_j at the tangent plane, and denote by $\mathbf{w}$ the projection of the point $\mathbf{v}$ from A_j at the tangent plane (We shall use same notation for corresponding vectors). From the geometry of sphere, it follows that

$$\sup_{\substack{\mathbf{v}_1, \mathbf{v}_2 \in A_j \\ j=1,2,...,N(\epsilon)}} \frac{\|\mathbf{w_1} - \mathbf{w_2}\|}{\|\mathbf{v_1} - \mathbf{v_2}\|} \geqslant 1 - 2\epsilon.$$

So, from (1) it follows that for all sufficiently small ϵ there exist $\delta(\epsilon) > 0$ such that for every $j = 1, 2, ..., N(\epsilon)$, for the covariance function $r_j((t, \mathbf{w}_1), (s, \mathbf{w}_2))$ of the Gaussian field $Z_j(t, \mathbf{w}) = Y(t, \mathbf{v})$, $\mathbf{v} \in A_j$ the following holds true,

$$(1-2\epsilon)\left(|t-s|^\alpha + \frac{1}{2}||\mathbf{w}_1 - \mathbf{w}_2||^2\right) \le 1 - r_j((t, \mathbf{w}_1), (s, \mathbf{w}_2)) \le (1+2\epsilon)\left(|t-s|^\alpha + \frac{1}{2}||\mathbf{w}_1 - \mathbf{w}_2||^2\right),$$

where $t, s \in [0, \delta(\epsilon)]$, $\mathbf{w}_1, \mathbf{w}_2 \in A_j^P$. Partitioning the interval $L_u = [0, h_u T]$ onto intervals of the length $\delta(\epsilon)$ and using that Y is stationary with respect to t, we get that the last sum in (7) does not exceed

$$\frac{2\mu(u)^{-1}T}{\delta(\epsilon)} \sum_{j=1}^{N(\epsilon)} \mathbf{P}\left(\max_{([0,\delta(\epsilon)]\times A_j)\cap \mathcal{R}_b} Y(t, \mathbf{v}) \le u, \max_{[0,\delta(\epsilon)]\times A_j} Y(t, \mathbf{v}) > u\right) =$$

$$\frac{2\mu(u)^{-1}T}{\delta(\epsilon)} \sum_{j=1}^{N(\epsilon)} \mathbf{P}\left(\max_{([0,\delta(\epsilon)]\times A_j^P)\cap \mathcal{R}^j(b)} Z_j(t, \mathbf{w}) \le u, \max_{[0,\delta(\epsilon)]\times A_j^P} Z_j(t, \mathbf{w}) > u\right). \tag{8}$$

Let $\xi(t, \mathbf{w})$, $t \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^{n-1}$, be a stationary Gaussian field with zero mean and covariance function $\rho(t, \mathbf{w}) = e^{-|t|^\alpha - \frac{1}{2}||\mathbf{w}||^2}$. Below we use the following operation of multiplication a vector by a set. Let A be a set in $\mathbb{R}^n$ and $\mathbf{b}$ be a vector in $\mathbb{R}^n$ with positive coordinates. Then we set $\mathbf{b}A := (\mathbf{x} = (x_1, ..., x_n) : (x_1/b_1, ..., x_n/b_n) \in A)$. Further, by Slepian's inequality, (8) can be estimated from above by

$$\frac{2\mu(u)^{-1}T}{\delta(\epsilon)} \sum_{j=1}^{N(\epsilon)} \Bigg\{\mathbf{P}\left(\max_{\mathbf{c}(-\epsilon)[([0,\delta(\epsilon)]\times A_j^P)\cap \mathcal{R}^j(b)]} \xi(t, \mathbf{w}) \le u, \max_{\mathbf{c}(-\epsilon)[([0,\delta(\epsilon)]\times A_j^P)]} \xi(t, \mathbf{w}) > u\right) +$$

$$\left[\mathbf{P}\left(\max_{\mathbf{c}(-\epsilon)[([0,\delta(\epsilon)]\times A_j^P)]} \xi(t, \mathbf{w}) \le u\right) - \mathbf{P}\left(\max_{\mathbf{c}(\epsilon)[([0,\delta(\epsilon)]\times A_j^P)]} \xi(t, \mathbf{w}) \le u\right)\right]\Bigg\} \tag{9}$$

where $\mathbf{c}(\pm\epsilon) = ((1 \pm 2\epsilon)^{1/\alpha}, (1 \pm 2\epsilon)^{1/2}, ..., (1 \pm 2\epsilon)^{1/2})$ and ϵ is sufficiently small. First we estimate the second sum in the right-hand part of (9). By Lemma 6.1 [3], for sufficiently large u, one can find constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$

such that the sum can be estimated by

$$\frac{2\mu(u)^{-1}T}{\delta(\epsilon)} C_1 u^{n+2/\alpha-1}\Psi(u)\delta(\epsilon)\cdot\epsilon\sum_{j=1}^{N(\epsilon)} V(A_j^P) \le C_2\epsilon,$$

where V denotes the volume in corresponding dimension. Turn to the first sum in the right-hand part of (9). Denote

$$\mathbf{g}_u = ((1-2\epsilon)^{1/\alpha}u^{-2/\alpha}, (1-2\epsilon)^{1/2}u^{-1}, ..., (1-2\epsilon)^{1/2}u^{-1}),$$

$S = bN$, N is integer, $K = [0,S]^n$. Since ξ is stationary, we have, partitioning the set $\mathbf{c}(-\epsilon)[([0,\delta(\epsilon)] \times A_j)]$ onto parallelepipeds, equal to $\mathbf{g}_u K$, that the first sum can be estimated by

$$(10)\qquad \frac{2\mu(u)^{-1}T}{S^n u^{1-n-2/\alpha}}\sum_{j=1}^{N(\epsilon)} \mathbf{P}\left(\max_{\mathbf{g}_u[K\cap\mathcal{R}^j(b)]}\xi(t,\mathbf{w}) \le u, \max_{\mathbf{g}_u K}\xi(t,\mathbf{w}) > u\right) V(A_j^P).$$

Let $\theta(t,\mathbf{w})$, $t\in\mathbb{R}$, $\mathbf{w}\in\mathbb{R}^{n-1}$ be a Gaussian separable field with parameters

$$\mathbf{E}\theta(t,\mathbf{w}) = (1-2\epsilon)(|t|^\alpha + \frac{1}{2}||\mathbf{w}||^2),$$
$$\mathbf{Var}(\theta(t,\mathbf{w}_1) - \theta(s,\mathbf{w}_2)) = 2(1-2\epsilon)(|t-s|^\alpha + \frac{1}{2}||\mathbf{w}_1 - \mathbf{w}_2||^2).$$

Following evaluations in the proof of Lemma 6.1 from [3] one can show that

$$\lim_{u\to\infty}\sqrt{2\pi}ue^{u^2/2}\mathbf{P}\left(\max_{\mathbf{g}_u[K\cap\mathcal{R}^j(b)]}\xi(t,\mathbf{w}) \le u, \max_{\mathbf{g}_u K}\xi(t,\mathbf{w}) > u\right)$$
$$= \int_0^\infty e^s\mathbf{P}\left(\max_K \theta(t,\mathbf{w}) > s, \max_{K\cap b\mathbb{Z}^n}\theta(t,\mathbf{w}) \le s\right) ds.$$

Since trajectories of θ are a.s. continuous, the probability under the integral tends to zero as $b\to 0$, for any s (for a fixed $S = Nb$). By Theorem of dominated convergences, last integral tends to zero when $b\to 0$. Thus for sufficiently large u and sufficiently small b, the first sum in the right-hand part of (10) can be bounded by ϵ, so Lemma follows. ∎

Let $L\in\mathcal{L}$ and T is as large as $L\subset[0,T]$. Let δ be a positive number. We partition the interval $[0,h_uT]$ onto intervals of length one intermittent with intervals of length δ. We denote by N, the number of all the intervals with length one and by π_u, the union of all the intervals. Denote also by λ_u, the union of all the intervals with length one, which are contained in L_u. We will use the notation $\mathcal{R}(b) = bu^{-2/\alpha}\mathbb{Z}$ in the one-dimension case as well.

Lemma 4. *For any $L \in \mathcal{L}$ with $\mu(L) > 0$ and every $\varepsilon > 0$ one can find $\delta > 0$ such that for all sufficiently large u, $P_\chi(u, \lambda_u \cap \mathcal{R}(b)) - P_\chi(u, L_u \cap \mathcal{R}(b)) \leq \varepsilon$.*

Proof. Let $L \subset [0, T]$, using (4) for all sufficiently large u, we have,

$$P_\chi(u, \lambda_u \cap \mathcal{R}(b)) - P_\chi(u, L_u \cap \mathcal{R}(b)) = \mathbf{P}\left(\max_{\lambda_u \cap \mathcal{R}(b)} \chi(t) \leq u, \max_{L_u \cap \mathcal{R}(b)} \chi(t) > u\right)$$

$$\leq \overline{P}_\chi(u, L_u \setminus \lambda_u) \leq 2\frac{\mu(u)^{-1}T}{1+\delta}\overline{P}_\chi(u, [0, \delta]) \leq \frac{\delta T}{1+\delta} \leq \varepsilon,$$

by obvious appropriate choice of δ. The Lemma is proved. ∎

Denote by $K_j,\ j = 1, 2, ...$ the intervals of length one from the π_u, recall that they are alternated by intervals of length δ. Consider infinitely many independent copies $Y_j(t, \mathbf{v})$ of the Gaussian field $Y(t, \mathbf{v})$, $(t, \mathbf{v}) \in K_j \times S_{n-1}$, $j = 1, 2, ...$ and introduce the Gaussian random field $Y_0(t, \mathbf{v}) = Y_j(t, \mathbf{v})$ for $(t, \mathbf{v}) \in K_j \times S_{n-1}$.

Lemma 5. *For any $L \in \mathcal{L}$,*

$$P_Y(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b) - P_{Y_0}(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b) \to 0 \text{ as } u \to \infty.$$

Proof. By Berman's inequality, we have,

$$|P_Y(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b) - P_{Y_0}(u, (\lambda_u \times S_{n-1}) \cap \mathcal{R}_b)|$$
$$\leq \frac{1}{\pi}\sum_{i \neq j} \sum_{\substack{(t,\mathbf{v}_1) \in (K_i \times S_{n-1}) \cap \mathcal{R}_b \\ (s,\mathbf{v}_2) \in (K_j \times S_{n-1}) \cap \mathcal{R}_b}} |r_Y((t, \mathbf{v}_1), (s, \mathbf{v}_2))|$$
$$\times \int_0^1 (1 - hr_Y((t, \mathbf{v}_1), (s, \mathbf{v}_2))^{-1/2} \exp\left(-\frac{u^2}{1 + hr_Y((t, \mathbf{v}_1), (s, \mathbf{v}_2))}\right) dh,$$

where r_Y is the covariance function of the field Y, r_{Y_0} is the covariance function of the field Y_0. Now we are in a position to estimate the last sum to show that it tends to zero. To begin with, note that from the equality

$$r_Y((t, \mathbf{v}_1), (s, \mathbf{v}_2)) = r(t - s)\left(1 - \frac{1}{2}||\mathbf{v}_1 - \mathbf{v}_2||^2\right)$$

and (3) it follows that there is γ_2, $0<\gamma_2<1$ such that

$$|r_Y((t,\mathbf{v}_1),(s,\mathbf{v}_2))| \le 1-\gamma_2$$

as $|t-s|\ge\delta$. Further, consider first "not too outstanding" t and s, that is, $t\in K_i$, $s\in K_j$, and $d(K_i,K_j)\le\mu(u)^{-\gamma_1}$, where $\gamma_1\in(0,1+\frac{1}{2}\gamma_2)$. (We define $d(K_i,K_j):=\sup\{|t-s|: t\in K_i, s\in K_j\}$.) Denoting by Σ_1 the part of the sum over such $t,s,\mathbf{v}_1,\mathbf{v}_2$ we get,

$$\begin{aligned}\Sigma_1 &\le \mathcal{C}_1 \sum_{\substack{i\neq j\\ d(K_i,K_j)\le\mu(u)^{-\gamma_1}}} \sum_{\substack{(t,\mathbf{v}_1)\in(K_i\times S_{n-1})\cap\mathcal{R}_b\\ (s,\mathbf{v}_2)\in(K_j\times S_{n-1})\cap\mathcal{R}_b}} \exp\left(-\frac{u^2}{2}(1+\frac{\gamma_2}{2})\right)\\ &= O\left(\mu(u)^{-1}\epsilon^{\frac{n(n-1)}{2}}u^{n-1+2/\alpha}\mu(u)^{-\gamma_1}u^{n-1+2/\alpha}\epsilon^{\frac{n(n-1)}{2}}e^{-\frac{u^2}{2}(1+\frac{\gamma_2}{2})}\right)=o(1)\end{aligned}$$

as $u\to\infty$, where $\mathcal{C}_1$ is a constant. We have used above that the volume of every A_j has order $\epsilon^{\frac{n(n-1)}{2}}$, for small ϵ.

Turn now to those t,s, $\mathbf{v}_1,\mathbf{v}_2$ for which $d(K_i,K_j)\ge\mu(u)^{-\gamma_1}$, $t\in K_i$, $s\in K_j$. Denote the corresponding part of the sum by Σ_2. From (2) we get in this case,

$$\sup_{|t-s|\ge\mu(u)^{-\gamma_1},\mathbf{v}_1,\mathbf{v}_2\in S_{n-1}} r_Y((t,\mathbf{v}_1),(s,\mathbf{v}_2)) := \kappa(u)=o(u^{-2})$$

as $u\to\infty$, so that

$$\begin{aligned}\Sigma_2 &\le \mathcal{C}_2\kappa(u) \sum_{\substack{i\neq j\\ d(K_i,K_j)\ge\mu(u)^{-\gamma_1}}} \sum_{\substack{(K_i\times S_{n-1})\cap\mathcal{R}_b\\ (K_j\times S_{n-1})\cap\mathcal{R}_b}} \exp\left(-\frac{u^2}{1+|r_Y((t,\mathbf{v}_1),(s,\mathbf{v}_2))|}\right)\\ &= O\left(\left(\mu(u)^{-1}\epsilon^{\frac{n(n-1)}{2}}u^{n-1+2/\alpha}\right)^2\kappa(u)e^{-u^2}\right)=O(u^2\kappa(u))=o(1)\end{aligned}$$

as $u\to\infty$, where $\mathcal{C}_2$ is a constant. Thus the lemma is proved. ■

P r o o f o f T h e o r e m 1. Note that Lemma 3 holds true also for the field Y_0, with the same grid. Let N be the number of unit intervals in the λ_u. It is easily to see that for any $\varepsilon>0$ one can chose δ sufficiently small in order to have $|N\mu(u)-V(L)|\le\varepsilon$, $V(\cdot)$ is now the length. From here we get,

$$P_{Y_0}(u,\lambda_u\times S_{n-1})=(1-\overline{P}_{Y_0}(u,\lambda_u\times S_{n-1}))^N\to e^{-V(L)},\ \text{as } u\longrightarrow\infty.$$

Taking into account Lemma 2, Lemma 3, Lemma 4, Lemma 5, we obtain the first relation in (5). It is easy to see that the second assertion of (5) follows from the equivalence of $\mu(u)$ and $\mu_A(u)$. Now the proof of Theorem 1 is completed.

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