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STS(21) with Automorphisms of Order 3 with 3 Fixed Points and 7 Fixed Blocks [1]

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The Steiner triple systems of order 21 with automorphisms of order 3 with 3 fixed points and 7 fixed blocks are constructed. There are 2963 nonisomorphic designs which are classified with respect to resolvability and the order of their automorphism group. Among them there are 28 resolvable designs yielding 40 Kirkman triple systems of order 21.

Key Words: combinatorial design, Steiner triple system, classification, automorphism

1. Introduction

For the basic concepts and notations concerning combinatorial designs refer to [1], [2], [12] and [14].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ – a finite collection of k-element subsets of V, called *blocks*. We say that $D = (V, \mathcal{B})$ is a *design* with parameters t-(v,k,λ), if any t-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

The incidence matrix of a design is a (0,1) matrix of v rows and b columns, where the element of the i-th row and j-th column equals 1, if $P_i \in B_j$ ($i = 1, 2, ..., v$, $j = 1, 2, ..., b$) and 0 if not. The design is completely determined by its incidence matrix.

Two designs are *isomorphic* if their incidence matrices are equivalent.

An *automorphism* of the design is a permutation of the points that transforms the blocks into blocks. All the automorphisms of a design form a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

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A *resolution* is a partition of the blocks into subsets called *parallel classes* such that each point is in exactly one block of each parallel class.

One of the most important properties of a design is its resolvability. The design is *resolvable* if it has at least one resolution.

A Steiner triple system of order v (denoted STS(v)) is a 2-(v,3,1) design (see for instance [4] and [5]). All the Steiner triple systems of orders $v < 19$ have been classified [4]. Recently Östergård and Kaski completed the classification of STS(19) [7], and thus STS(21) are now the Steiner triple systems with smallest parameters that have still not been classified, or at least enumerated.

Wilson showed in 1974 that the nonisomorphic STS(21) with three subsystems STS(7) are at least 2160980 [15]. Recently Kaski, Östergård, Topalova and Zlatarski [8] showed that the exact number of these designs is 2166351 and presented their full classification. They yield 13036 nonisomorphic Wilson Kirkman triple systems.

Classifications of some other smaller classes of STS(21) have been done by several authors, i.e. Mathon, Phelps and Rosa [9], [10], Tonchev [13], Kapralov and Topalova [6]. These works lead to the construction of 2049 nonisomorphic STS(21) (108 of them of Wilson type), and 215 KTS(21)(56 of Wilson type). The classification of STS(21) with some automorphisms of order 3, with any automorphisms of order 2, and without automorphisms is still an open problem.

The resolutions (if such exist) of an STS(v) are called Kirkman triple systems of order v (KTS(v)). The Kirkman triple systems of order 21 with nontrivial automorphisms have already been constructed [3].

This paper offers a classification of all the STS(21) with automorphisms of order 3 with 3 fixed points and 7 fixed blocks. Construction tips are presented in section 2, and the classification results are summarized in section 3.

2. Construction

Let D be a 2-(21,3,1) design with an automorphism φ of order 3 fixing 3 points and 7 blocks. We can permute the points and blocks of the design in such a way that points/blocks of one and the same orbit become neighbours. Thus without loss of generality we can assume that φ acts as (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19)(20)(21) on the points, and as (1,2,3)(4,5,6) ... (61,62,63)(64)(65)(66)(67)(68)(69)(70) on the blocks. Then the incidence matrix of D is of the form:

$$
\begin{array}{cccccccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} & a_{1,10} & \dots & a_{1,21} & i\ o\ o\ o\ o\ o\ o \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} & a_{2,10} & \dots & a_{2,21} & o\ i\ o\ o\ o\ o\ o \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} & a_{3,10} & \dots & a_{3,21} & o\ o\ i\ o\ o\ o\ o \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & a_{4,9} & a_{4,10} & \dots & a_{4,21} & o\ o\ o\ i\ o\ o\ o \\
a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} & a_{5,9} & a_{5,10} & \dots & a_{5,21} & o\ o\ o\ o\ i\ o\ o \\
a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & a_{6,9} & a_{6,10} & \dots & a_{6,21} & o\ o\ o\ o\ o\ i\ o \\
iii & iii & iii & ooo & ooo & ooo & ooo & ooo & ooo & ooo & \dots & ooo & 0\ 0\ 0\ 0\ 0\ 0\ 1 \\
ooo & ooo & ooo & iii & iii & iii & ooo & ooo & ooo & ooo & \dots & ooo & 0\ 0\ 0\ 0\ 0\ 0\ 1 \\
ooo & ooo & ooo & ooo & ooo & ooo & iii & iii & iii & ooo & \dots & ooo & 0\ 0\ 0\ 0\ 0\ 0\ 1
\end{array}
$$

where $a_{i,j}$, $i = 1, 2, \dots, 6$, $j = 1, 2, \dots, 21$ are circulant matrices of order 3, $iii = (1, 1, 1)$, $ooo = (0, 0, 0)$, $i = iii^t$ and $o = ooo^t$.

Let $m_{i,j}$, $i = 1, 2, 3, 4, 5, 6$, $j = 1, 2, \dots, 21$ denote the number of 1's in a row of $a_{i,j}$. The following equations hold for the matrix $M = (m_{i,j})_{6\times 21}$

(1) $$\sum_{j=1}^{21} m_{i,j} = 9, \qquad i = 1, 2, \dots, 6$$

(2) $$\sum_{j=1}^{3} m_{i,j} = 1, \quad \sum_{j=4}^{6} m_{i,j} = 1, \quad \sum_{j=7}^{9} m_{i,j} = 1, \qquad i = 1, 2, \dots, 6$$

(3) $$\sum_{j=1}^{21} m_{i,j}^2 = 9, \qquad i = 1, 2, \dots, 6$$

(4) $$\sum_{j=1}^{21} m_{i_1,j} m_{i_2,j} = 3, \qquad 1 \le i_1 < i_2 \le 6.$$

It follows from (1) and (3) that $m_{i,j}$ can only be 0 or 1. Taking into consideration (2) and (4) we can see that M can be uniquely extended to an incidence matrix of a 2-(7,3,3) design by adding the row which starts with nine 1-s followed by twelve 0-s.

There are 10 nonisomorphic 2-(7,3,3) designs [11]. By cutting a row of each of these 10 incidence matrices in all the possible ways, 21 inequivalent solutions for M were obtained. In 20 of these solutions the columns which have two ones can be partitioned into three groups of three columns (see (2)) in a unique way. Only for one of the matrices there are two possibilities to consider. The 1-s in the 22 matrices thus obtained were replaced in all the possible ways by circulant matrices of order 3 with one 1 in each row/column. Then the fixed part was added, and thus 2963 2-(21,3,1) designs were constructed.

3. Classification

Three of the designs (with an order of the automorphism group equal to 9) possess an automorphism of order 3 with no fixed points and 7 fixed blocks, and have already been constructed in [6], and there are 28 resolvable designs which have been constructed in [3]. The other 2932 designs are new.

Table 1. presents a classification of the constructed designs with respect to the order of their full automorphism group.

Table 1: Full automorphism group order of the constructed designs

Aut.Gr.Order	3	6	9	12	18	27	36	54	108	Total
Designs	2868	46	25	4	14	3	1	1	1	2963

The base blocks of the designs which have an order of the automorphism group greater than 9 are presented in Table 2, where the fixed blocks are omitted. The order of the automorphism group is given in the column denoted by *Aut*.

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Table2: Base blocks of some designs

Des	B_1	B_4	B_7	B_{10}	B_{13}	B_{16}	B_{19}	B_{22}	B_{25}	B_{28}	B_{31}	B_{34}	B_{37}	B_{40}	B_{43}	B_{46}	B_{49}	B_{52}	B_{55}	B_{58}	B_{61}	*Aut*
	1	7	13	1	7	13	1	7	13	1	1	1	1	1	1	4	4	4	4	4	4	
1	4	10	16	6	12	18	5	11	17	7	9	8	10	12	11	7	9	8	10	12	11	108
	19	19	19	20	20	20	21	21	21	13	14	15	16	17	18	16	17	18	13	14	15	
	1	7	13	1	7	13	1	7	13	1	1	1	1	1	1	4	4	4	4	4	4	
2	4	10	16	6	12	18	5	11	17	7	9	8	10	12	11	7	9	8	10	12	11	18
	19	19	19	20	20	20	21	21	21	13	14	15	16	17	18	18	16	17	13	14	15	
	1	7	13	1	7	13	1	7	13	1	1	1	1	1	1	4	4	4	4	4	4	
3	4	10	16	6	12	18	5	11	17	7	9	8	10	12	11	7	9	8	10	12	11	54
	19	19	19	20	20	20	21	21	21	13	14	15	18	16	17	17	18	16	13	14	15	
	1	7	13	1	7	13	1	7	13	1	1	1	1	1	1	4	4	4	4	4	4	
4	4	10	16	6	12	18	5	11	17	7	9	8	10	12	11	7	9	8	10	11	12	18
	19	19	19	20	20	20	21	21	21	13	14	18	15	16	17	14	18	16	13	15	17	
	1	7	13	1	7	13	1	7	13	1	1	1	1	1	1	4	4	4	4	4	4	
5	4	10	16	6	12	18	5	11	17	7	9	8	12	10	11	7	9	8	10	12	11	36
	19	19	19	20	20	20	21	21	21	13	14	16	15	18	17	14	16	17	15	13	18	
	1	7	13	1	7	13	1	7	10	1	1	1	1	1	1	4	4	4	4	4	4	
6	4	10	16	6	12	18	5	13	16	7	9	8	10	12	13	7	8	9	10	12	11	12
	19	19	19	20	20	20	21	21	21	11	18	16	14	15	17	14	13	16	15	17	18	
	1	7	13	1	7	13	1	7	10	1	1	1	1	1	1	4	4	4	4	4	4	
7	4	10	16	6	12	17	5	13	17	7	9	8	10	12	13	7	8	9	10	12	11	12
	19	19	19	20	20	20	21	21	21	11	16	17	14	15	18	14	13	17	15	18	16	
	1	7	13	1	7	10	1	7	10	1	1	1	1	1	1	4	4	4	4	4	4	
8	4	10	16	6	13	16	5	16	13	7	9	8	10	11	15	7	8	9	10	12	14	18
	19	19	19	20	20	20	21	21	21	12	13	18	14	16	17	11	13	17	15	16	18	
	1	7	13	1	7	10	1	7	10	1	1	1	1	1	1	4	4	4	4	4	4	
9	4	10	16	6	13	16	5	16	13	7	9	8	10	11	13	7	8	9	10	12	13	18
	19	19	19	20	20	20	21	21	21	12	14	18	15	16	17	11	15	17	14	16	18	
	1	7	13	1	7	10	1	7	10	1	1	1	1	1	1	4	4	4	4	4	4	
10	4	10	16	6	13	16	5	16	13	7	9	8	11	10	14	7	8	9	12	10	15	18
	19	19	19	20	20	20	21	21	21	12	13	16	15	17	18	11	13	16	14	18	17	
	1	4	13	1	7	10	7	10	1	1	1	1	1	1	1	4	4	4	4	4	7	
11	7	10	16	4	13	16	14	18	6	5	9	8	12	13	15	7	9	8	12	11	12	18
	19	19	19	20	20	20	21	21	21	17	10	11	14	18	16	15	17	18	13	14	16	
	1	4	13	1	7	10	7	10	1	1	1	1	1	1	1	4	4	4	4	4	7	
12	7	10	16	4	13	16	14	18	6	5	9	8	11	13	14	9	7	8	12	11	11	18
	19	19	19	20	20	20	21	21	21	18	12	10	15	17	16	14	16	18	15	13	18	
	1	4	13	1	4	10	7	1	10	1	1	1	1	1	1	4	4	4	4	7	7	
13	10	7	16	7	13	16	16	4	13	6	5	8	9	11	13	8	9	12	11	10	15	18
	19	19	19	20	20	20	21	21	21	14	16	12	15	18	17	10	17	14	16	14	17	
	1	4	13	1	4	10	7	1	10	1	1	1	1	1	1	4	4	4	4	7	7	
14	10	7	16	7	13	16	16	4	13	6	5	8	9	11	15	8	9	12	10	12	15	18
	19	19	19	20	20	20	21	21	21	14	17	12	13	18	16	11	17	14	18	13	17	

Table 2 (continued): Base blocks of some designs

Des	B_1	B_4	B_7	B_{10}	B_{13}	B_{16}	B_{19}	B_{22}	B_{25}	B_{28}	B_{31}	B_{34}	B_{37}	B_{40}	B_{43}	B_{46}	B_{49}	B_{52}	B_{55}	B_{58}	B_{61}	Aut
	1	4	13	1	4	10	7	1	10	1	1	1	1	1	1	4	4	4	4	7	7	
15	10	7	16	8	13	17	16	4	13	6	5	9	7	12	14	8	9	10	11	10	13	18
	19	19	19	20	20	20	21	21	21	13	16	11	15	17	18	12	16	15	17	14	18	
	1	4	13	1	4	10	7	1	10	1	1	1	1	1	1	4	4	4	4	7	7	
16	10	7	16	8	13	17	16	4	13	6	5	9	7	12	15	8	9	10	12	11	13	18
	19	19	19	20	20	20	21	21	21	13	18	11	14	17	16	11	16	15	18	15	18	
	1	7	10	1	7	10	1	7	10	1	1	1	1	1	1	4	4	4	4	4	4	
17	4	13	16	6	15	18	5	18	13	7	9	8	13	15	14	7	9	8	10	12	11	12
	19	19	19	20	20	20	21	21	21	10	11	12	18	16	17	14	16	17	15	13	18	
	1	7	10	1	7	10	1	7	10	1	1	1	1	1	1	4	4	4	4	4	4	
18	4	13	16	6	15	17	5	16	13	7	9	8	13	15	14	7	9	8	10	12	11	12
	19	19	19	20	20	20	21	21	21	10	11	12	16	17	18	14	17	18	15	13	16	
	1	7	10	1	7	10	7	4	1	1	1	1	1	1	1	4	4	4	4	4	7	
19	4	16	13	6	17	14	14	18	10	5	9	8	12	15	14	8	7	10	12	14	11	18
	19	19	19	20	20	20	21	21	21	7	11	13	18	16	17	11	13	15	17	16	18	
	1	4	7	1	4	7	4	1	7	1	1	1	1	1	1	4	4	4	7	7	7	
20	10	16	13	12	18	15	10	13	16	4	6	5	11	15	14	12	11	15	12	10	11	
	19	19	19	20	20	20	21	21	21	7	8	9	18	16	17	13	14	17	14	18	17	18
	1	4	7	1	4	7	1	4	7	1	1	1	1	1	1	4	4	4	7	7	7	
21	10	13	16	15	17	10	18	10	14	4	6	5	12	11	14	12	11	15	12	11	13	27
	19	19	19	20	20	20	21	21	21	7	8	9	13	16	17	14	18	16	15	17	18	
	1	4	7	1	4	7	1	4	7	1	1	1	1	1	1	4	4	4	7	7	7	
22	10	13	16	13	16	10	18	12	15	4	6	5	12	11	15	11	10	15	12	11	14	27
	19	19	19	20	20	20	21	21	21	7	8	9	14	17	16	14	18	17	13	18	17	
	1	4	7	1	4	7	1	4	7	1	1	1	1	1	1	4	4	4	7	7	7	
23	10	13	16	15	17	10	17	12	13	4	6	5	12	11	14	11	10	15	12	11	15	27
	19	19	19	20	20	20	21	21	21	7	8	9	13	16	18	14	16	18	14	18	17	
	1	4	7	7	4	1	1	4	7	1	10	1	1	1	1	1	7	4	4	7	4	
24	13	16	10	18	15	12	17	11	14	15	13	4	6	5	9	11	12	8	10	11	13	18
	19	20	21	19	21	20	21	19	20	16	16	7	8	10	14	18	17	17	14	13	18	

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