Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 3-4

Bessel-Sobolev Type Spaces

Miloud Assal [1] *and Mohamed Mounir Nessibi* [2]

Presented at Internat. Congress "MASSEE' 2003", 4th Minisymposium "TMSF"

In this paper we investigate Sobolev type spaces denoted by $E^{s,p}_\alpha$ ($s \in I\!R$, $p \in [1,+\infty]$) associated to the Bessel-operators $L_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}$, on $I\!R_+$. We develop some basic properties of Sobolev spaces H^s_α obtained for $p = 2$ and stress out on the very important Sobolev imbedding and the corresponding compact imbedding which allows Reillich's theorem and Poincaré's inequality.

AMS Subj. Classification: 44A15, 46E35, 42A38, 42C40

Key Words: Sobolev-type spaces, Fourier-Bessel transform, Sobolev imbedding, Reillich's theorem, Poincaré's inequality

1. Introduction

The theory of classical Sobolev spaces on $I\!R^n$ [1] has been generalized on different measurable spaces using their corresponding Lebesgue spaces ([2], [7]). In this work, using the Fourier-Bessel transform, we define and study Bessel-Sobolev type spaces $E^{s,p}_\alpha$ ($s \in I\!R, p \in [1,+\infty]$) on $I\!R_+$. We recall that for a suitable function $f : [0,\infty[\longrightarrow \mathbb{C}$, the Fourier-Bessel transform of f is defined by ([8], p. 15):

$$(1) \qquad F(f)(\lambda) = \int_0^\infty j_\alpha(\lambda x) f(x) dm_\alpha(x), \quad \text{where} \quad dm_\alpha(x) = \frac{x^{2\alpha+1}dx}{2^\alpha \Gamma(\alpha+1)},$$

and j_α is the normalized Bessel function of first kind and order α [9]. The Fourier-Bessel transform is an isomorphism from the Schwartz subspace $S_*(I\!R)$ consisting of even functions into itself ([8], p.127). We denote by L_α ($\alpha > \frac{-1}{2}$) the operators

$$(2) \qquad L_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx},$$

and we recall that they satisfy the following properties [8].

1) $$L_\alpha(j_\alpha(x\lambda)) = -\lambda^2 j_\alpha(x\lambda), \text{ for all } x, \lambda \geq 0.$$

2) For a suitable function f we have

$$F(L_\alpha f)(\lambda) = -\lambda^2 F(f)(\lambda) \quad \text{and} \quad L_\alpha(Ff)(\lambda) = F(-x^2 f)(\lambda).$$

The Bessel-Sobolev type spaces $E_\alpha^{s,p}$ ($s \in I\!R$ and $p \in [1, +\infty]$) are given by :

$$E_\alpha^{s,p} = \left\{ T \in S'_*(I\!R); (1+\xi^2)^s F(T) \in L^p(dm_\alpha) \right\}$$

where $S'_*(I\!R)$ is the space of even tempered distributions on $I\!R$. It has been proved in [7] that the Bessel-Sobolev type spaces $E_\alpha^{s,p}$ endowed with the norm $||.||_{E_\alpha^{s,p}} = ||(1+\xi^2)^s F(.)||_{dm_\alpha}$ are complete and that $S_*(I\!R)$ is dense in $E_\alpha^{s,p}$ ($p \in [1, +\infty]$). It is also shown that the inclusion map $E_\alpha^{s,p} \subset E_\alpha^{t,p}; s > t$ is continuous.

In this work, we introduce the spaces H_α^s ($s \in I\!R$) obtained for $p = 2$ and establish a compactness type imbedding result, as well as a Reillich's theorem which allows to a Poincaré's inequality on these spaces.

Finally, we mention that C will be used to denote a constant which may vary from line to line.

2. Preliminaries

Let $\alpha > -\frac{1}{2}$ be a fixed real number. We equip the space $[0, \infty[$ with the measure $dm_\alpha(x)$ and by $L^p(dm\alpha)$ we denote the corresponding Lebesgue spaces endowed with the norms

$$\begin{aligned} \|f\|_{L^p(dm_\alpha)} &= \left(\int_0^\infty |f(x)|^p dm_\alpha(x) \right)^{\frac{1}{p}} ; \quad 1 \leq p < +\infty, \\ \|f\|_{L^\infty(dm_\alpha)} &= \operatorname*{ess\,sup}_{x \geq 0} |f(x)|. \end{aligned}$$

The generalized translation operators T_x^α ($x \geq 0$) associated with Bessel-operators are defined for a suitable function f by ([8], p.93)

$$T_x^\alpha f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f\left(\sqrt{x^2+y^2+2xy\cos\theta}\right) (\sin\theta)^{2\alpha} d\theta$$

and they satisfy the following properties:

1) $\Big[T_x^\alpha j_\alpha(\lambda.)\Big](y) = j_\alpha(\lambda x) j_\alpha(\lambda y), \quad$ for all $x, y,\ \lambda \geq 0$.

2) For all f in $L^p(dm_\alpha)$, $T_x^\alpha f$ belongs to $L^p(dm_\alpha)$ and satisfies the following inequality : ([8], p. 94)

$$\|T_x^\alpha f(y)\|_{L^p(dm_\alpha)} \leq \|f\|_{L^p(dm_\alpha)}.$$

3) For all f in $L^1(dm_\alpha)$, we have

$$F(T_x^\alpha f)(\lambda) = j_\alpha(\lambda x) F(f)(\lambda), \quad \text{for all } \lambda, x \geq 0.$$

4) For all $f \in S_*(I\!R)$, $T_x^\alpha f$ is given by ([8], p. 93)

$$T_x^\alpha f(y) = \int_0^\infty f(t) W_\alpha(x, y, t) t^{2\alpha+1} dt, \text{ for all } x, y > 0, \tag{3}$$

where $W_\alpha(x, y, .)$ is a kernel supported by $[|x-y|, x+y]$ and satisfying

$$\int_0^\infty W_\alpha(x, y, t) t^{2\alpha+1} dt = 1. \tag{4}$$

The convolution product of a pair of functions f and g is given by ([8], p. 97)

$$f * g(x) = \int_0^\infty T_x^\alpha f(y) g(y) dm_\alpha(y). \tag{5}$$

It has been proved in [8] that for all $p \geq 1$, $L^p(dm_\alpha)$ is dense in $S'_*(I\!R)$. Hence, for all $\psi \in L^p(dm_\alpha)$ and $f \in S_*(I\!R)$, $< \psi, f >$ means the value of $\psi \in S'_*(I\!R)$ on f and it is given by :

$$< \psi, f >= \int_0^\infty \psi(x) f(x) dm_\alpha(x).$$

It might be observed that a long list of properties of the classical distributions in $I\!R^n$ remains valid also in our context.

Let us now give some notations that will be used. For all $m \in I\!N$ and $f \in \mathcal{C}^\infty(I\!R)$ we denote by

- $\gamma_m(f) = \sup\limits_{\substack{q,p \leq m \\ x \geq 0}} \Big|(1+x^2)^p (\frac{d}{dx^2})^q f(x)\Big|$ where $\frac{d}{dx^2} = \frac{1}{x}\frac{d}{dx}$.
- $\mathcal{C}_*^\infty(I\!R)$ the subspace of $\mathcal{C}^\infty(I\!R)$ consisting of even functions.

Now, using the properties of the operator $\dfrac{d}{dx^2}$ given in [8] we obtain the following characterizations :

- $\mathcal{C}_*^\infty(I\!R) = \Big\{ f : I\!R \longrightarrow \mathbb{C} \ / \ \left(\frac{d}{dx^2}\right)^k \Big(f\Big) \in \mathcal{C}_*^1(I\!R), \forall k \in I\!N \Big\}.$
- $S_*(I\!R) = \Big\{ f \in \mathcal{C}_*^\infty(I\!R) / \gamma_m(f) < \infty, \forall\, m \in I\!N \Big\}.$

In what follows we give some general properties which may be easily verified by using the different properties of the Bessel-Fourier transform and the convolution product associated with Bessel operators [8].

P1) For all $m \in I\!N$ and $p \in [1, \infty]$. The Sobolev-Bessel type space $E_\alpha^{m,p}$ is consisting of the tempered distributions $T \in S'_*(I\!R)$ such that $F\left[(-L_\alpha)^j(T)\right]$ belongs to $L^p(dm_\alpha)$, for all $j \in \{0, 1, ..., m\}$.

P2) Let $s \in I\!R$, $p \in [1, \infty[$, $\varphi \in S_*(I\!R)$ and $T \in E_\alpha^{s,p}$. Then φT belongs to $E_\alpha^{s,p}$. Moreover, the mapping: $(\varphi, T) \longmapsto \varphi.T$ from $S_*(I\!R) \times E_\alpha^{s,p}$ into $E_\alpha^{s,p}$ is bilinear continuous.

P3) For all $p \in [1, \infty[$, $s \in I\!R$ and $k \in I\!N$, we have $(-L_\alpha)^k \Big(E_\alpha^{s,p}\Big) \subset E_\alpha^{s-k,p}$ with continuous imbedding.

P4) For all $m \in I\!N$, $E_\alpha^{m,2}$ is consisting of functions $f \in L^2(dm_\alpha)$ such that $(-L_\alpha)^j f \in L^2(dm_\alpha)$, for all $j \in \{0, 1, ..., m\}$. In particular $E_\alpha^{0,2} = L^2(dm_\alpha)$.

3. The space H_α^s

We now turn to the particular spaces $E_\alpha^{s,2}$ denoted by H_α^s. One of their basic properties is that, endowed with the inner product

$$(S, T)_{H_\alpha^s} = \int_0^\infty ((1 + \xi^2)^{2s} F(S)(\xi) \overline{F(T)(\xi)} dm_\alpha(x)$$

they become Hilbert spaces.

Proposition 3.1. *Let $m \in I\!N$. Then for all $s \geq \frac{\alpha+1}{2} + m$,*

$$H_\alpha^s \subset \mathcal{C}_*^m(I\!R).$$

In what follows we give a new characterization of H_α^{-s} for $s \in I\!N$.

Theorem 3.1. *Let $m \in I\!N$, then all elements T of H_α^{-m} can be written in the following form:*

$$T = \sum_{k=0}^{m} C_m^k (-L_\alpha)^k g, \text{ where } g \in L^2(dm_\alpha).$$

P r o o f. The result holds from the Plancherel's theorem. ∎

Theorem 3.2. *For all $s \in]0,1[$, the space H_α^s is characterized as follows:*

$$H_\alpha^s = \left\{ f \in L^2(dm_\alpha) \,/ \int_0^\infty \int_0^\infty \frac{|f(x) - T_u^\alpha f(x)|^2}{u^{1+4s}} dm_\alpha(x) du < \infty \right\}.$$

P r o o f. Remark that, for all $s > 0$, H_α^s is consisting of functions $f \in L^2(dm_\alpha)$ satisfying

$$\int_0^\infty \xi^{4s} |F(f)(\xi)|^2 dm_\alpha(\xi) < \infty.$$

Hence, using the fact that ([9])

$$1 - j_\alpha(t) \underset{t \to 0}{\sim} \frac{-t^2}{2\Gamma(\alpha+1)} \quad \text{and} \quad |j_\alpha(t)| \le 1, \quad \forall\, t \ge 0,$$

we obtain by Plancherel's theorem

$$\int_0^\infty \xi^{4s} |F(f)(\xi)|^2 dm_\alpha(\xi) = C \int_0^\infty \int_0^\infty \frac{|(f - T_u^\alpha f)(x)|^2}{u^{1+4s}} dm_\alpha(x) du$$

which gives the desired result. ∎

Proposition 3.2. *Let $\varphi \in S_*(I\!R)$. Then for all $s, t \in I\!R$ such that $t < s$, the operator $T \longmapsto \varphi.T$ from H_α^s into H_α^t is compact.*

P r o o f. Let $(T_n)_{n \in I\!N}$ be a sequence in H_α^s such that $||T_n||_{H_\alpha^s} \le 1$, then by Alaoglu's theorem ([3], p.42) there exists a subsequence $(T_{n_k})_{k \in I\!N}$ weakly converging to T in H_α^s . Put $v_k = T_{n_k} - T$, then for all $R > 0$, we have

$$||\varphi.v_k||_{H_\alpha^t}^2 \le \int_0^R (1+\xi^2)^{2t} |F(\varphi.v_k)(\xi)|^2 dm_\alpha(\xi) + \frac{||\varphi.v_k||_{H_\alpha^s}^2}{(1+R^2)^{2(s-t)}}.$$

Using P1) we get

$$\frac{||\varphi.v_k||_{H_\alpha^s}^2}{(1+R^2)^{2(s-t)}} \le \frac{C(\gamma_m(\varphi))^2 . \left(1 + ||T||_{H_\alpha^s}\right)^2}{(1+R^2)^{2(s-t)}}, \quad \text{for all } k \in I\!N.$$

Hence, for all $\varepsilon > 0$ and R sufficiently large, we obtain

$$(6) \qquad ||\varphi.v_k||^2_{H^t_\alpha} \leq \int_0^R (1+\xi^2)^{2t}|F(\varphi.v_k)(\xi)|^2 dm_\alpha(\xi) + \frac{\varepsilon}{2}, \quad \text{for all } k \in I\!N.$$

Now using the fact that $F(\varphi.v_k) = \left(v_k, F^{-1}\left[(1+x^2)^{-2s}T^\alpha_\xi \overline{F(\varphi)}\right]\right)_{H^s_\alpha} = 0$, it holds from Lebesgue's theorem $||\varphi.v_k||^2_{H^t_\alpha} < \varepsilon$, for all $\varepsilon > 0$. Then the result is proved. ■

Notation. Let $K \subset I\!R$ be a compact. We denote by $H^s_{\alpha,K}$; $s \in I\!R$ the subspace of H^s_α consisting of distributions T supported by K. We have the following theorem.

Theorem 3.3. **(Reillich's theorem)** *Let $s, t \in I\!R$; $t < s$. Then for all compact $K \subset I\!R$, the canonical imbedding $H^s_{\alpha,K} \hookrightarrow H^t_{\alpha,K}$ is compact.*

Proof. Let $\widetilde{K} = K \cup (-K)$ and let V a relatively compact neighborhood of K. So, by virtue of Urysohn's theorem ([4], p.237), there exists $\varphi \in \mathcal{D}(I\!R)$ such that $\varphi \equiv 1$ on $V \cup (-V)$. Put

$$\Psi(x) = \frac{\varphi(x) + \varphi(-x)}{2}, \text{ for all } x \in I\!R.$$

Then we obtain $\Psi.T = T$, for all $T \in H^s_{\alpha,K}$. The desired result holds from Proposition 3.2. ■

Corollary 3.1. *Let K be a compact included in $I\!R$. Then, for all $s \geq 0$, there exists $C > 0$ satisfying*

$$(7) \quad \frac{1}{C}||T||_{H^s_\alpha} \leq \left(\int_0^\infty \xi^{4s}|F(T)(\xi)|^2 dm_\alpha(\xi)\right)^{\frac{1}{2}} \leq C.||T||_{H^s_\alpha}, \quad \forall T \in H^s_{\alpha,K}.$$

Proof. Suppose that there is no constant $C > 0$ checking the left hand side inequality of (7). Then, for all $k \in I\!N$ there exists $T_k \in H^s_{\alpha,K}$ satisfying

$$\frac{1}{k}||T_k||_{H^s_\alpha} > \left(\int_0^\infty \xi^{4s}|F(T_k)(\xi)|^2 dm_\alpha(\xi)\right)^{\frac{1}{2}}.$$

Without loss of generality we can suppose that $||T_k||_{H^s_\alpha} = 1$. So it holds

$$\lim_{k\to\infty} \int_0^\infty \xi^{4s}|F(T_k)(\xi)|^2 dm_\alpha(\xi) = 0.$$

By Reillich's theorem we deduce that there exists a subsequence $(T_{k_p})_p$ of $(T_k)_k$ converging to T in $L^2(dm_\alpha)$, and so, Hölder's inequality leads to

$$||T_{k_p} - T||_{L^1(dm_\alpha)} \leq C_K ||T_{k_p} - T||_{L^2(dm_\alpha)}$$

which implies that $\lim_{k\to\infty} ||F(T_{k_p}) - F(T)||_\infty = 0$, since (see [8], p. 139) $||F(T_{k_p}) - F(T)||_\infty \leq ||T_{k_p} - T||_{L^1(dm_\alpha)}$. So, we obtain

$$\int_0^\infty |\xi|^{4s} |F(T)(\xi)|^2 dm_\alpha(\xi) = 0,$$

and hence $T = 0$. On the other hand, by Plancherel's theorem we obtain

$$1 = ||T_{k_p}||^2_{H^s_\alpha} \leq C||T_{k_p}||^2_{L^2(dm_\alpha)} + C\int_0^\infty \xi^{4s} |F(T_{k_p})(\xi)|^2 dm_\alpha(\xi) \,\forall p \in I\!N$$

which leads to an absurdity by tending p to ∞.

To complete the proof, it suffice to verify that

$$\frac{1}{1+C}||T||_{H^s_\alpha} \leq \left(\int_0^\infty \xi^{4s} |F(T)(\xi)|^2 dm_\alpha(\xi)\right)^{\frac{1}{2}} \leq (1+C)||T||_{H^s_\alpha}.$$

Then the result comes out. ■

Theorem 3.4. **(Poincaré's inequality)** *Let $s, t \in I\!R$; $0 \leq t \leq s$. Then there exists $C > 0$ satisfying*

$$||T||_{H^t_\alpha} \leq C\varepsilon^{2(s-t)} ||T||_{H^s_\alpha}, \quad \textit{for all } \varepsilon > 0 \textit{ and } T \in H^s_{\alpha,\varepsilon} = H^s_{\alpha,[-\varepsilon,\varepsilon]}.$$

Proof. Let $T \in H^s_{\alpha,\varepsilon}$ and $f \in S_*(I\!R)$. Put $< T_\varepsilon, f >= \frac{1}{\varepsilon^{2\alpha+2}} < T, f_{\frac{1}{\varepsilon}} >$ where $f_{\frac{1}{\varepsilon}}(x) = f\left(\frac{x}{\varepsilon}\right)$. Then T_ε belongs to $H^s_{\alpha,1}$ and we have

$$\int_0^\infty \xi^{4t} |F(T_\varepsilon)(\xi)|^2 dm_\alpha(\xi) \leq C\int_0^\infty \xi^{4s} |F(T_\varepsilon)(\xi)|^2 dm_\alpha(\xi), \quad \forall T \in H^s_{\alpha,\varepsilon}.$$

On the other hand using the fact that $F(T_\varepsilon)(\xi) = \frac{1}{\varepsilon^{2\alpha+2}} F(T)(\frac{\xi}{\varepsilon})$, we obtain

$$\varepsilon^{2t}\left(\int_0^\infty \eta^{4t} |F(T)(\eta)|^2 dm_\alpha(\eta)\right)^{\frac{1}{2}} \leq C\varepsilon^{2s}\left(\int_0^\infty \eta^{4s} |F(T)(\eta)|^2 dm_\alpha(\eta)\right)^{\frac{1}{2}}.$$

Thus, the desired result holds by virtue of Corollary 3.1. ■

References

[1] R. A. Adams. *Sobolev Spaces*, Academic Press, N. York-San Francisco-London, 1975.

[2] N. Ben Salem, A. Dachraoui. Sobolev type spaces associated with Jacobi differential operators, *Integral Transforms and Special Functions* **9**, No 3, 2000, 163-184.

[3] H. Bresis. *Analyse fonctionnelle: (Théorie et applications)*, Masson, Paris, 3ème tirage, 1992.

[4] G. B. Folland. *Real Analysis (Modern Techniques and Their Applications),* A Wiley-Intersci. Publication, N. York-Chichester-Brisbane-Toronto-Singapore, 1984.

[5] N. M. Lebedev. *Special Functions and Their Applications*, Dover, N. York, 1972.

[6] F. W. J. Olver. *Asymptotics and Special Functions,* Academic Press, N. York, 1974.

[7] R. S. Pathak, P. K. Pandey. Sobolev type spaces associated with Bessel operators, *J. of Mathematical Analysis and Applications* **215**, 1997, 95-111.

[8] K. Trimeche. *Generalized Harmonic Analysis and Wavelet Packets*, Gordon And Breach Science Publ., Australia.

[9] G. N. Watson. *A Treatise of the Theory of Bessel Functions*, Second Ed., Cambridge Univ. Press, London-N. York, 1966.

[1] *Département de Mathématiques IPEIN*
Campus Universitaire
Mrezka, 8000 Nabeul, TUNISIA
e-mail: Miloud.Assal@fst.rnu.tn

Received: 30.09.2003

[2] *Département de Mathématiques, Faculté des Sciences de Bizerte*
Zarzouna, 7021 Bizerte, TUNISIA
e-mail: Mounir.Nessibi@fsb.rnu.tn