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A Tauberian Theorem of Littlewood Type for the Summation by Means of Laguerre Polynomials

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For the summations of divergent series, defined by means of the classical Laguerre polynomials, a Tauberian theorem of Litllewood type is proved.

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1. Introduction

The series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \dots \tag{1.1}$$

is called $(P-A)$-summable, if the series $\sum_{n=0}^{\infty} a_n z^n$ converges in the disk $\{z \in \mathbb{C} : |z| < 1\}$ and, moreover, there exists

$$\lim_{z \to 1-0} \sum_{n=0}^{\infty} a_n z^n = s \neq \infty. \tag{1.2}$$

By definition, the complex number s is called $(P-A)$-sum of the series (1.1).

The $(P-A)$-summation is regular. It means that if the series (1.1) is convergent, then it is $(P-A)$-summable, and its $(P-A)$-sum is equal to its sum in the usual sense.

The $(P-A)$-summability of the series (1.1) does not imply, in general, its convergence. But if $a_n = O(n^{-1})$ when $n \to \infty$, i.e., if the sequence $\{na_n\}_{n=0}^{\infty}$ is bounded, and the series (1.1) is $(P-A)$-summable, then it is convergent. The last assertion is known as the Tauberian theorem of Littlewood [1, **7.6.6**].

2. Series in Laguerre polynomials

We define the Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with parameter $\alpha \neq -1, -2, -3, \ldots$ by their Rodrigues type formula

$$L_n^{(\alpha)} = \frac{1}{n!} \frac{1}{z^\alpha \exp(-z)} \frac{d^n}{dz^n} z^{n+\alpha} \exp(-z), \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

provided that $z \in \mathbb{C} \setminus [0, \infty)$.

Let $0 < \lambda < \infty$ and define $\Delta(\lambda) = \{z \in \mathbb{C} : \Re(-z)^{1/2} < \lambda\}$, and $\Delta^*(\lambda) = \mathbb{C} \setminus \overline{\Delta(\lambda)}$. It is easy to see that $\Delta(\lambda)$ is the interior of the parabola $p(\lambda)$ with focus at the origin and vertex at the point $-\lambda^2$, and that $\Delta^*(\lambda)$ is the exterior of $p(\lambda)$. Further, we assume that $\Delta(0) = \emptyset, \Delta(\infty) = \mathbb{C}, \Delta^*(0) = \mathbb{C} \setminus [0, \infty)$, and $\Delta^*(\infty) = \emptyset$.

Proposition 1. *Let $\{a_n\}_{n=0}^{\infty}$ be an arbitrary sequence of complex numbers, and define*

$$\lambda = \max\{0, -\limsup_{n \to \infty} (2\sqrt{n})^{-1} \log |a_n|\}. \tag{2.2}$$

If $\alpha \neq -1, -2, -3, \ldots$ is real, then the series

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z) \tag{2.3}$$

is absolutely uniformly convergent on every compact subset of the region $\Delta(\lambda)$, and diverges in $\Delta^*(\lambda)$ [2, Theorem 4.3, b].

Remarks.

(1) The equality (2.2) can be regarded as a formula of Cauchy-Hadamard type for the series in Laguerre polynomials.

(2) In the proof of Proposition 1 it is used a particular case of Perron's asymptotic formula for the Laguerre polynomials with real parameter α [3, (8.22.3)]. Namely, the representation $(n \geq 1)$

$$L_n^{(\alpha)}(z) = l^{(\alpha)}(z) n^{\alpha/2 - 1/4} \exp\{2(-z)^{1/2}\sqrt{n}\}\{1 + \lambda_n^{(\alpha)}(z)\} \tag{2.4}$$

holds in the region $\mathbb{C} \setminus [0, \infty)$, where

$$l^{(\alpha)}(z) = (2\sqrt{\pi})^{-1}(-z)^{-\alpha/2 - 1/4} \exp(z/2), \tag{2.5}$$

$\{\lambda_n^{(\alpha)}(z)\}_{n=1}^{\infty}$ is a sequence of functions holomorphic in the region $\mathbb{C} \setminus [0, \infty)$ and, moreover,

$$\lambda_n^{(\alpha)}(z) = O(n^{-1/2}) \tag{2.5}$$

uniformly on every compact subset of this region.

3. $(L^{(\alpha)},\zeta)$-summation

Let $\alpha > -1, \zeta \in \mathbb{C} \setminus [0,\infty)$, and define

$$\mathcal{L}_n^{(\alpha)}(z;\zeta) = \{L_n^{(\alpha)}(z)\}\{L_n^{(\alpha)}(\zeta)\}^{-1}, \quad n = 0,1,2,\ldots. \tag{3.1}$$

Let us note that the system of polynomials $\{\mathcal{L}_n^{(\alpha)}(z;\zeta)\}_{n=0}^{\infty}$ is well-defined since, because of the assumption that $\alpha > -1$, all the zeros of Laguerre polynomials with parameter α are in the interval $(0,\infty)$ and, hence, $L_n^{(\alpha)}(\zeta) \neq 0$ for every $\zeta \in \mathbb{C} \setminus [0,\infty)$ and $n = 0,1,2,\ldots$.

The series (1.1) is called $(L^{(\alpha)},\zeta)$-summable if the series

$$\sum_{n=0}^{\infty} a_n \mathcal{L}_n^{(\alpha)}(z;\zeta) \tag{3.2}$$

is convergent in the region $\Delta(\lambda)$ with $\lambda = \Re(-\zeta)^{1/2}$ and, moreover, there exists

$$\lim_{\delta \to +0} \sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(\zeta+\delta;\zeta) \neq \infty. \tag{3.3}$$

Remark. Every $(L^{(\alpha)},\zeta)$-summation is regular [4].

In [5] is given a result of Tauberian type for the $(L^{(\alpha)},\zeta)$-summations, namely:

Proposition 2. *If the series (1.1) is $(L^{(\alpha)},\zeta)$-summable, and*

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \ldots + na_n}{n+1} = 0, \tag{3.4}$$

then it is convergent. In particular, this holds if $a_n = o(n^{-1})$ when $n \to \infty$.

4. The main result

The main result in this paper is a Littlewood type generalization of the $o(n)$-version of Proposition 2 in the case when $\zeta = -\lambda^2$.

Proposition 3. *Let $\alpha >= 1$ and $0 < \lambda < \infty$. If the series (1.1) is $(L^{(\alpha)}, -\lambda^2)$-summable, and $a_n = O(n^{-1})$, $n \to \infty$, then it is convergent.*

Proof. We assume that $a_0 = 0$ which is not an essential restriction. Then as a corollary of the asymptotic formula (2.4) as well as of (2.5) we obtain that for $\delta \in [0, \lambda^2)$,

$$\text{(3.5)} \qquad \mathcal{L}_n^{(\alpha)}(-\lambda^2+\delta;-\lambda^2)$$
$$= D^{(\alpha)}(\lambda,\delta)\exp\{-2(\lambda-\sqrt{\lambda^2-\delta})\sqrt{n}\}\{1+\gamma_n^{(\alpha)}(\lambda,\delta)\},$$

where $D^{(\alpha)}(\lambda,\delta)\neq 0$ for $\delta\in[0,\lambda^2), \lim_{\delta\to 0} D^{(\alpha)}(\lambda,\delta)=1, \gamma_n^{(\alpha)}(\lambda,0)=0$, and $\gamma_n^{(\alpha)}(\lambda,\delta)=O(n^{-1/2})$ uniformly with respect to δ in any interval $[0,\theta\lambda^2]$ with $\theta\in(0,1)$.

We choose $\theta=1/2$ and define

$$F^{(\alpha)}(\lambda,\delta)=F_1^{(\alpha)}(\lambda,\delta)+F_2^{(\alpha)}(\lambda,\delta),\quad \delta\in[0,\lambda^2/2],$$

where

$$F_1^{(\alpha)}(\lambda,\delta)=D^{(\alpha)}(\lambda,\delta)\sum_{n=1}^{\infty}a_n\exp\{-2(\lambda-\sqrt{\lambda^2-\delta})\sqrt{n}\},$$

and

$$\text{(3.7)} \qquad F_2^{(\alpha)}(\lambda,\delta)=D^{(\alpha)}(\lambda,\delta)\sum_{n=1}^{\infty}a_n\exp\{-2(\lambda-\sqrt{\lambda^2-\delta})\sqrt{n}\}\gamma_n^{(\alpha)}(\lambda,\delta).$$

Obviously, the assumption that the series (1.1) is $(L^{(\alpha)},-\lambda^2)$-summable implies the existence of

$$\text{(3.8)} \qquad \lim_{\delta\to 0}F^{(\alpha)}(\lambda,\delta)\neq\infty.$$

Further, it is easily seen that the series in the right-hand side of equality (3.7) is uniformly convergent with respect to $\delta\in[0,\lambda^2/2]$ and, hence, there exists $\lim_{\delta\to 0}F_2^{(\alpha)}(\lambda,\delta)\neq\infty$. Finally, from (3.6) it follows that if we denote $u=2(\lambda-\sqrt{\lambda^2-\delta})$, then there exists

$$\lim_{u\to+0}\sum_{n=1}^{\infty}a_n\exp(-u\sqrt{n})=\lim_{\delta\to 0}\sum_{n=1}^{\infty}a_n\exp\{-2(\lambda-\sqrt{\lambda^2-\delta})\sqrt{n}\}\neq\infty.$$

But $a_n=O(n^{-1})$ implies

$$a_n=O\Big(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}}\Big),\ n\to\infty,$$

and by Theorem 104 from [6], it follows that the series (1.1) is convergent. ∎

5. Comments

It seems that the statement of Proposition 3 holds for every $(L^{(\alpha)}, \zeta)$-summation. This will be really the fact if every such a summation is equivalent, e.g., to the $(L^{(\alpha)}, -\lambda^2)$-summation with $\lambda = \Re(-\zeta)^{1/2}$ but this is still an open problem.

It is interesting whether the $(L^{(\alpha)}, -\lambda^2)$-summation of the series (1.1) with real a_n, $n = 0, 1, 2, \ldots$, and the assumption that $a_n > -Cn^{-1}$, $C = Const.$, imply its convergence. If this is true, it will be a Tauberian theorem of Landau type for the $(L^{(\alpha)}, -\lambda^2)$-summations.

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