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On Finite-integrable p-Analytic Partial Differential Equations

Miloš Čanak [1], *Liljana Stefanovska* [2], *Ljubomir Protić* [3]

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Some classes of p-analytic partial differential equations allowing a closed form general solution are studied. Application of obtained results to some mechanical problems is given.

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1. Introduction

The generalized analytic functions of the first and second class are introduced by Položij [1] by the following definitions:

Definition 1.1. The function $f(z,\bar{z}) = u + iv$ is a generalized analytic function of the first class (or p-analytic function with $p = p(x,y)$ characteristics) in the domain T if it is defined and continuous in T together with u'_x, u'_y, v'_x, v'_y satisfying

$$u'_x = \frac{1}{p}v'_y, \quad u'_y = -\frac{1}{p}v'_x. \tag{1.1}$$

Definition 1.2. The function $f(z,\bar{z}) = u + iv$ is a generalized analytic function of the second class (or (p,q)-analytic function with $p = p(x,y)$ and $q = q(x,y)$ characteristics) in the domain T if it is defined and continuous in T together with u'_x, u'_y, v'_x, v'_y satisfying

$$pu'_x - qu'_y = v'_y, \quad qu'_x + pu'_y = -v'_x. \tag{1.2}$$

The system (1.1) can be transformed to the Vekua differential equation that is solved by Vekua [2], but the solution cannot be used because it contains infinite series and double singular integrals with Cauchy kernel, which generally cannot be expressed in a closed form. If the general solution of (1.1), or the general solution of the corresponding p-analytic complex differential equation is in closed, finite and explicit form

$$f = f(z, \bar{z}, Q(z), \overline{Q(z)}\,), \quad (f = u + iv), \tag{1.3}$$

where $Q(z)$ is an arbitrary analytic function, then it is possible to separate its real and imaginary part. Because there are no such examples in the available literature, in this paper we give some results for finite-integrable p-analytic differential equations that can be solved in a closed form.

2. Some finite-integrable p-analytic complex differential equations

a) **An equation with constant characteristics.** The system of partial differential equations (1.1) can be represented in the form

$$u'_x - v'_y = \frac{1-p}{p} v'_y, \;\; u'_y + v'_x = -\frac{1-p}{p} v'_x. \tag{2.1}$$

By linear combination of equations (2.1), the following p-analytic complex differential equation is obtained

$$p f'_{\bar{z}} + i(1-p) v'_{\bar{z}} = 0 \;\; (p \neq 0, f(z, \bar{z}) = u + iv). \tag{2.2}$$

In the special case, when $p = c = \text{const}$, the equation (2.2) becomes

$$[pf + i(1-p)v]'_{\bar{z}} = 0, \tag{2.3}$$

wherefrom it follows

$$pf + i(1-p)v = Q(z) \quad \text{or} \quad \frac{p+1}{2} f + \frac{p-1}{2} \overline{f} = Q(z), \tag{2.4}$$

where $Q(z)$ is an arbitrary analytic function. We consider the conjugated equation as well

$$\frac{p+1}{2} \overline{f} + \frac{p-1}{2} f = \overline{Q(z)}. \tag{2.5}$$

The solution of the system made by the second equation from (2.4) and (2.5) is

$$f(z, \bar{z}) = \frac{p+1}{2p} Q(z) - \frac{p-1}{2p} \overline{Q(z)}. \tag{2.6}$$

By this equation, a p-analytic function with a real constant characteristics is explicitly expressed via an arbitrary analytic function $Q(z)$ and the characteristics $p=c$.

b) **An equation with characteristics** $p=x^2$**.** By the substitution $v=v_1p$, where v_1 is a new unknown function, the system (2.1) is transformed into

$$u'_x - v'_{1y} = \frac{1}{p}(p'_y v_1), \quad u'_y + v'_{1x} = -\frac{1}{p}(p'_x v_1),$$

which transforms by linear combination into

$$f'_{\bar z} = -\frac{p'_{\bar z}}{2p}(f-\overline{f}), \quad (f = u+iv_1). \tag{2.7}$$

Let us choose $p=x^2=\left(\frac{z+\bar z}{2}\right)^2$ and $p'_{\bar z}=\frac{z+\bar z}{2}$. Then the equation (2.7) becomes

$$f'_{\bar z} = -\frac{f-\overline{f}}{z+\bar z}. \tag{2.8}$$

As the right-hand side of (2.8) is imaginary, the left-hand side must be imaginary too. It is shown by Fempl [3] that $f'_{\bar z}$ is an imaginary if and only if the function f is

$$f = \psi'_y + i\psi'_x, \tag{2.9}$$

where $\psi=\psi(x,y)$ is an arbitrary real $\mathbb{C}^2$ function. Then, $f'_{\bar z}=\frac{i}{2}(\psi''_{xx}+\psi''_{yy})$, and the equation (2.8) is transformed into the second order real elliptic partial differential equation

$$\psi''_{xx} + \psi''_{yy} + \frac{2}{x}\psi'_x = 0 \tag{2.10}$$

Our idea is to present the general solution of the equation (2.10) in terms of an arbitrary harmonic function. By the substitution $\psi=v/x$, where $v=v(x,y)$ is a new unknown real function, the equation (2.10) is transformed into

$$v''_{xx} + v''_{yy} = 0. \tag{2.11}$$

The general solution of (2.11) is $v=h(x,y)$, and the general solution of (2.10) is

$$\psi(x,y) = \frac{h(x,y)}{x}, \tag{2.12}$$

where $h(x,y)$ is an arbitrary harmonic function. The equation (2.12) can be written in complex form as

$$\psi(x,y) = \frac{h(x,y)}{x} = \frac{(Q+\overline{Q})/2}{(z+\bar z)/2} = \frac{Q+\overline{Q}}{z+\bar z}, \quad Q=Q(z)=h(x,y)+ih_1(x,y)$$

which reduces to

$$f(z,\bar z) = 2i\frac{Q'(z)(z+\bar z)-(Q+\overline{Q})}{(z+\bar z)^2},$$

a general solution of the differential equation (2.7), where $Q = Q(z)$ is an arbitrary analytic function. By separating real and imaginary part of the solution, one gets

$$u(x,y) = -\frac{q'_{2x}}{x}, \quad v_1(x,y) = \frac{xq'_{1x}-q_1}{x^2}, \quad (Q(z) = q_1 + iq_2).$$

As $v = x^2 v_1$, then

$$u(x,y) = -\frac{q'_{2x}}{x}, \quad v(x,y) = xq'_{1x} - q_1, \tag{2.13}$$

which is the general solution of the system

$$u'_x = \frac{1}{x^2}v'_y, \quad u'_y = -\frac{1}{x^2}v'_x, \tag{2.14}$$

expressed via an arbitrary harmonic function $q_1(x,y)$.

c) **An equation with characteristics** $p = 1/4x^2$. By the substitution $f = Up^{-1/2}$, where $U = U(z,\bar z)$ is a new unknown function, the complex equation (2.7) is transformed into

$$U'_{\bar z} = \frac{p'_{\bar z}}{2p}\overline{U}. \tag{2.15}$$

Let choose $p = \frac{1}{(z+\bar z)^2} = \frac{1}{4x^2}$ to be the characteristics, then the equation (2.15) is transformed into

$$U'_{\bar z} = -\frac{1}{z+\bar z}\overline{U}. \tag{2.16}$$

The general solution of equation (2.16) according to [4] is

$$U = Q'(z) - \frac{Q+\overline{Q}}{z+\bar z}, \tag{2.17}$$

where $Q(z)$ is an arbitrary analytic function. Now the above substitution gives the system

$$u = 2xq'_{1x} - 2q_1, \quad v_1 = -2xq'_{1y} \text{ or } u'_x = 4x^2v'_y, \quad u'_y = -4x^2v'_x \tag{2.18}$$

with the general solution ($q_1 = q_1(x,y)$ is an arbitrary harmonic function)

$$u = 2xq'_{1x} - 2q_1, \quad v = -q'_{1y}/2x. \tag{2.19}$$

d) **An equation with characteristics** $p = x^k$. The p-analytic functions with characteristics $p = x^k$ are of especial importance. Substituting $p = x^k$ and $p'_{\bar{z}} = \frac{kx^{k-1}}{2}$, (2.15) becomes

$$U'_{\bar{z}} = \frac{k}{2(z+\bar{z})}\overline{U}. \tag{2.20}$$

The equation (2.20) is known as the Vekua complex differential equation. Using Mitrinović method [4], one particular solution is obtained as $U_p = (z+\bar{z})^{k/2}$.

e) **An equation with characteristics** $p = e^{h(x,y)}$. Similarly, we consider p-analytic functions with the characteristics $p = e^{h(x,y)}$, where $h(x,y)$ is an arbitrary harmonic function. For $h(x,y) = Q(z) + \overline{Q(z)}$, ($Q(z)$ is an arbitrary analytic function), it obtains

$$\frac{p'_{\bar{z}}}{2p} = \frac{e^{Q+\overline{Q}} \cdot \overline{Q'_{\bar{z}}}}{2e^{Q+\overline{Q}}} = \frac{\overline{Q'_{\bar{z}}}}{2} = \frac{\overline{a'(z)}}{\overline{a(z)}}, \quad (a(z) = e^{Q(z)/2}).$$

According to Mitrinović method, it is possible to find a particular solution of the equation $U'_{\bar{z}} = \dfrac{\overline{a'(z)}}{\overline{a(z)}}$ as $U_p = a(z)\overline{a(z)} = e^{\frac{Q+\bar{Q}}{2}}$.

3. On a vector field for p-analytic functions

Now the following questions may be risen: when and under which conditions the general solution exists and how to find it. The vector field for p-analytic functions gives a partial answer. Bilimović [5] has developed a geometrical theory of nonanalytic complex functions $w(z,\bar{z}) = u(x,y) + iv(x,y)$, based on the concept of the *middle derivative*

$$\mu(w) = \frac{1}{2}[u'_x + v'_y + i(v'_x - u'_y)] = \mu_1 + i\mu_2, \tag{3.1}$$

and *deviation from analytic*

$$B(w) = (u'_x - v'_y) + i(u'_y + v'_x) = B_1 + iB_2. \tag{3.2}$$

Complex functions (3.1) and (3.2) can be expressed in vector forms as $\vec{\mu}$ and $\vec{B}$ consequently. By using the operators $\vec{\mu}$ and $\vec{B}$ it is possible to express a directional derivative of nonanalytic function in vector form

$$\overrightarrow{w^{(\theta)}} = \vec{\mu} + \vec{\beta}e^{-2\theta i}, \quad \vec{\beta} = \frac{1}{2}\vec{B}. \tag{3.3}$$

To express the system of partial equations (1.1), which define p-analytic functions in a geometric form, we consider the following matrix identity

$$(3.4)\quad \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u'_x + v'_y & u'_y - v'_x \\ v'_x - u'_y & u'_x + v'_y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} u'_x - v'_y & v'_x + u'_y \\ v'_x + u'_y & -u'_x + v'_y \end{pmatrix}.$$

On the base of (3.1), (3.2) and (3.4) it is found

$$(3.5)\qquad u'_x = \mu_1 + B_1,\ \ u'_y = -\mu_2 + B_2,\ \ v'_x = \mu_2 + B_2,\ \ v'_y = \mu_1 - B_1.$$

By substitution of (3.5) in (1.1), it is obtained $\vec{\beta} = K\vec{\mu}$, where $K = (1-p)/(1+p)$, which is a vector form for p-analytic function (1.1). From the relation

$$(3.6)\qquad \vec{B} = \operatorname{grad} v_1 + \vec{k} \times \operatorname{grad} v_2 = (v'_{1x} - v'_{2y})\vec{\imath} + (v'_{1y} + v'_{2x})\vec{\jmath}$$

and from the system of partial equations

$$(3.7)\quad v'_{1x} - v'_{2y} = a(x,y)v_1 + b(x,y)v_2, \quad v'_{1y} + v'_{2x} = b(x,y)v_1 - a(x,y)v_2$$

obtained from the equation $V'_{\bar{z}} = \frac{p'_{\bar{z}}}{2p}\vec{V}$ where $V = v_1 + iv_2$, $a = \frac{p'_x}{2p}$, $b = \frac{p'_y}{2p}$ it follows

$$(3.8)\qquad \operatorname{div} \vec{B} = \nabla^2 v_1, \quad \operatorname{rot} \vec{B} = \vec{k}\nabla^2 v_2.$$

The following theorems treat classification of p-analytic functions.

Theorem 3.1. *The vector field of $\vec{B}$ for the system* (3.7) *is a Laplace field* ($\operatorname{div} \vec{B} = 0, \operatorname{rot} \vec{B} = \vec{0}$) *if and only if $p = c =$const.*

P r o o f. The sufficiency is trivial. To show the necessity, we will differentiate the first equation from (3.7) on x and the second on y and adding, which in combination with $v''_{1xx} + v''_{1yy} = 0$ gives

$$(3.9)\qquad a(v'_{1x} - v'_{2y}) + b(v'_{2x} + v'_{1y}) + v_1(a'_x + b'_y) + v_2(b'_x - a'_y) = 0.$$

Similar procedure in combination with $v''_{2xx} + v''_{2yy} = 0$, yields

$$(3.10)\qquad a(v'_{1y} + v'_{2x}) + b(v'_{2y} - v'_{1x}) + v_1(a'_y - b'_x) + v_2(b'_y + a'_x) = 0.$$

By substitution of $v'_{1x} - v'_{2y}$ and $v'_{2x} + v'_{2y}$ from (3.7) in (3.9) and (3.10) and after some calculations and respectively, $a(x,y) = p'_x/2p$, $\ b(x,y) = p'_y/2p$, the following homogeneous linear system on variables $pp''_{xx} + p\,p''_{yy}$ and ${p'_x}^2 + {p'_y}^2$ is obtained

$$2(pp''_{xx} + pp''_{yy}) - ({p'_x}^2 + {p'_y}^2) = 0, \quad 2(pp''_{xx} + pp''_{yy}) - 3({p'_x}^2 + {p'_y}^2) = 0.$$

Since the system has non-zero determinant, only the trivial solutions are possible, i.e. $pp''_{xx} + pp''_{yy} = 0$, ${p'_x}^2 + {p'_y}^2 = 0$, equivalent to $p = c = \text{const}$. ■

R e m a r k 3.1. In a special case when $c = 1$ the system (1.1), (similarly and (3.7)) transforms into $u'_x = v'_y, \quad u'_y = -v'_x$, and p-analytic functions are transformed into ordinary analytic functions.

Theorem 3.2. *The vector field of for the system of equations* (3.7) *is a solenoidal field* ($\mathrm{div}\vec{B} =$, $\mathrm{rot}\vec{B} \neq \vec{0}$) *if and only if* $p(x, y) = [c_1 h(x, y) + c_2]^2$, *where h = h(x, y) is an arbitrary harmonic function.*

P r o o f. As in the previous proof we will prove only the necessity of the condition. Now, *div* $\vec{B} = \nabla^2 v_1$ leads to (3.9) and to $(a^2 + b^2 + a'_x + b'_y)v_1 + (b'_x - a'_y)v_2 = 0$, which results into

$$2pp''_{xx} + 2pp''_{yy} - {p'_x}^2 - {p'_y}^2 = 0. \tag{3.11}$$

Now we are looking for a general solution of the equation (3.11) in the form $p = f(h)$, where f is an unknown $\mathbb{C}^2$ function and $h = h(x, y)$ is an arbitrary harmonic function. Then the equation (3.11) transforms into

$$(2ff'' - f'^2)({h'_x}^2 + {h'_y}^2) = 0. \tag{3.12}$$

The second term ${h'_x}^2 + {h'_y}^2$ vanishes if and only if $h(x, y) = \text{const}$ letting the vector field of $\vec{B}$ to be the Laplace field. Then must be $2ff'' - f'^2 = 0$, which general solution is $f = (c_1 h + c_2)^2$. This gives the general solution of (3.11) as $p(x, y) = [c_1 h(x, y) + c_2]^2$, where $h(x, y)$ is an arbitrary harmonic function, and c_1, c_2 are arbitrary real constants. ■

Theorem 3.3. *The vector field of* $\vec{B}$ *for the system of equations* (3.7) *is a potential field* ($\mathrm{div}\vec{B} \neq 0$, $\mathrm{rot}\vec{B} = \vec{0}$) *if and only if* $p(x, y) = [c_1 h(x, y) + c_2]^{-2}$, *where* $h = h(x, y)$ *is an arbitrary harmonic function.*

P r o o f. Starting by $\mathrm{div}\vec{B} = \nabla^2 v_1 \neq 0$ and $\vec{B} = \vec{k}\nabla^2 b_2 = \vec{0}$, the equation $v''_{2xx} + v''_{2yy} = 0$ is obtained. Repetition of the procedure from the proof of Theorem 3.2, yields

$$2pp''_{xx} + 2pp''_{yy} - 3{p'_x}^2 - 3{p'_y}^2 = 0$$

with the general solution $p(x, y) = [c_1 h(x, y) + c_2]^{-2}$. ■

The remaining case of complex field, $\mathrm{div}\vec{B} = \nabla^2 v_1 \neq 0$, $\mathrm{rot}\vec{B} = \vec{k}\nabla^2 v_2 \neq \vec{0}$, corresponds to the general form of the system (3.7) or (1.1). Then it is possible to use general consideration from the Polozhyi's monograph [1], or to use the complex form (2.15) and use the Vekua theory of complex differential equations [2].

4. Relations between the characters of vector field and finite-differentiable p-analytic differential equation

In this chapter the existence of connection between the type of the vector field and finite-integrability of the corresponding p-analytic differential equation is shown.

Laplace field. In the Chapter 3 it is shown that a constant characteristics $p = c$ corresponds to p-analytic function for Laplace field. The p-analytic function is explicitly expressed via formula (2.6), i.e. the solution of the system (1.1) with whom corresponds the p-analytic differential equation (2.3) expressed via an arbitrary analytic function $Q(z)$. By separating the real and imaginary part one obtains $u = \dfrac{1}{c}q_1$, $v = q_2$, $(Q(z) = q_1 + iq_2)$. It can be concluded that the system (1.1) is finite-integrable if the vector field for p-analytic function is Laplace one.

Solenoidal field. This vector field has characteristics $p(x, y) = [c_1 h(x, y) + c_2]^2$, where c_1 and c_2 are arbitrary real constants. Let it be $c_1 = 1$, $c_2 = 0$ and $p = h^2$, where $h(x, y)$ is an arbitrary harmonic function. The harmonic function can be expressed through $p = h^2 = [\varphi(z) = \overline{\varphi(z)}]^2$, where $\varphi(z)$ is an arbitrary analytic function. As

$$\frac{p'_{\bar z}}{2p} = \frac{2(\varphi + \overline{\varphi})\overline{\varphi'}}{2(\varphi + \overline{\varphi})^2} = \frac{\overline{\varphi'}}{\varphi + \overline{\varphi}},$$

the p-analytic differential equation (2.15) transforms into

$$U'_{\bar z} = \frac{\overline{\varphi'}}{\varphi + \overline{\varphi'}}\overline{U}, \tag{4.1}$$

where $\varphi(z)$ is an arbitrary, given function. By using the method of Mitrinović [4], one particular solution is $U_p = \dfrac{-i}{\varphi + \overline{\varphi}}$. Let the general solution of the equation (4.1) be found in the form

$$U(z, \bar z) = A(z)Q'(z) + U_p\Big[Q(z) + \overline{Q(z)}\Big],$$

where $A(z)$ is unknown analytic function and $Q(z)$ is an arbitrary analytic function. By substitution $\overline{U}$ and $U'_{\bar z}$ into (4.1), it is obtained $A = \frac{i}{\varphi'}$. Finally, the general solution of the differential equation (4.1) is

$$U(z, \bar z) = i\frac{Q'(z)}{\varphi'(z)} - i\frac{Q(z) + \overline{Q(z)}}{\varphi(z) + \overline{\varphi(z)}}. \tag{4.2}$$

It is possible to conclude that in the case of a solenoidal field for p-analytic functions, the corresponding p-analytic differential equations are finite-integrable.

Potential field. Similarly, as it was shown for solenoidal fields, the p-analytic differential equation is determined by

$$U'_{\bar{z}} = -\frac{\overline{\varphi'}}{\varphi + \overline{\varphi}}\overline{U}, \tag{4.3}$$

where $\varphi(z)$ is an arbitrary given function. On the same way, it is shown that the general solution of the equation (4.3) is

$$U(z,\bar{z}) = \frac{Q'(z)}{\varphi'(z)} - \frac{Q(z) + \overline{Q(z)}}{\varphi(z) + \overline{\varphi(z)}},$$

where $Q(z)$ is an arbitrary analytic function. It is possible to conclude also, that in the case of potential vector field the corresponding p-analytic differential equations are finite-integrable.

Complex field. It is possible to construct examples of finite-integrable p-analytic differential equations that correspond to the complex vector field of the vector $\vec{B}$. It means that for this type of field is not possible to answer the question of the finite-integrability by the vector analysis methods, but the other types of methods are necessary.

Remark 4.1. For (p,q) analytic functions, introduced by the definition 1.2, the similar considering is possible.

5. Conclusion

Except the theoretical meaning, finite-integrable p-analytic DE and corresponding systems of PDE (1.1) have many applications. 1. Among others, we will stress on the applications in mechanics (filtration flow, rotation body torsion, tension of rotational shell, axial-symmetry elastic theory etc), where real and imaginary part of p-analytic function may have the meaning of moving and tension. So, the solution of the system (1.1) and its corresponding p-analytic complex differential equation in finite and explicit form is of great importance, because only in that case the separation of a real and imaginary part with the clear physical meaning is possible. 2. The finite-integrable p-analytic differential equations are of great importance in the theory of p-polyanalytic functions. Čanak [6] shown how a non-analytic and n-times differentiable function $W(z,\bar{z})$ can be approximated with a p-polyanalytic function (polynomial with a p-polianalytic coefficients) of the n-th order and estimated the error. 3. The presentation of the solution of the system (1.1) in a finite form is of importance in the process of solving boundary value problems. The simplest case is Laplace field in which the classical problems of Riemann, Hilbert and Carleman directly are transformed to boundary value problems for analytic functions. More complex cases are solenoid and potential vector fields, and the other types of finite-integrable p-analytic functions, by the possibility of differentials of an unknown function in the boundary conditions. These arguments give a motivation of a developing a complete and a general theory of finite-integrable p-analytic differential equations and systems of PDE (1.1).

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[1] *Faculty of Agriculture*
University of Belgrade
SERBIA and MONTENEGRO

[2] *Faculty of Technology and Metallurgy*
Skopje, R. MACEDONIA
e-mail: liljana@ian.tmf.ukim.edu.mk

[3] *Faculty of Mathematics*
University of Belgrade
SERBIA and MONTENEGRO