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Extension of the Duhamel Principle for Time-Nonlocal Boundary Value Problems

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The classical Duhamel principle for evolution equation is generalized to the case when the initial value conditions of the form $u(x,0)=0$ and $u_t(x,0)=0$ are replaced by non-local conditions of the form $\chi_\tau\{u(x,\tau)\}=0$ and $\chi_\tau\{u_t(x,\tau)\}=0$, where χ is an arbitrary linear functional.

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1. Introduction

The classical Duhamel principle is usually stated separately both for the heat equation and the wave equation.

For uniformity, we consider the equations $u_t = u_{xx}$ and $u_{tt} = u_{xx}$ in the half strip $0 \le x \le a$, $0 \le t$. On the left-hand side of it we consider the usual boundary condition of the first kind $u(0,t)=0$ and instead of a local boundary value condition on the right-hand side we take a general non-local boundary value condition

$$\Phi_\xi\{u(\xi,t)\} = f(t),$$

where Φ is a non-zero continuous linear functional on $C^1[0,a]$, i.e. $\Phi \in (C^1[0,a])^*$.

For the heat equation $u_t = u_{xx}$ we consider a general initial value condition

(1) $$\chi_\tau\{u(x,\tau)\} = 0$$

with a continuous non-zero linear functional χ on $C[0,\infty)$. As it is well known, each such functional has a Riesz-Markov representation by Stieltjes integral

$$\chi_\tau\{\varphi(\tau)\} = \int_0^T \varphi(\tau)d\alpha(\tau)$$

with some $T < \infty$ and $\alpha \in BV[0,T]$.

In the case $\chi\{\varphi\} = \varphi(0)$ we have the classical initial value condition $u(x,0) = 0$.

For the wave equation we $u_{tt} = u_{xx}$ we consider two initial-value conditions

(2) $$\chi_\tau\{u(x,\tau)\} = 0$$

(3) $$\chi_\tau\{u_t(x,\tau)\} = 0.$$

The following two theorems generalize the classical Duhamel principle.

Theorem 1. *Let $\Omega(x,t)$ be a solution for one of the above boundary value problems for the special choice $f(t) \equiv 1$, the remaining conditions being homogeneous. Then*

$$u(x,t) = \frac{\partial}{\partial t}\chi_\tau\left\{\int_\tau^t \Omega(x,t+\tau-\sigma)f(\sigma)d\sigma\right\}$$

is a solution of the same problem, but for arbitrary $f \in C[0,\infty)$ when the other conditions remain homogeneous.

Theorem 2. *Let $\Omega(x,t)$ be a solution for the heat equation $u_t = u_{xx}$ with the boundary value conditions $u(0,t) = 0$, $\Phi_\xi\{u(\xi,t)\} = 1$ and the "initial" value condition $\chi_\tau\{u(x,\tau)\} = 0$. Then the function*

(4) $$u(x,t) = \frac{\partial}{\partial t}\Phi_\xi\chi_\tau\left\{\int_0^\xi h(x,t;\eta,\tau)d\eta\right\},$$

where

$$h(x,t;\eta,\tau) = -\frac{1}{2}\int_x^\xi\int_\tau^t \Omega(\xi+x-\eta,t+\tau-\sigma)F(\eta,\sigma)d\eta d\sigma$$

$$+\frac{1}{2}\int_{-x}^\xi\int_\tau^t \Omega(|\xi-x-\eta|,t+\tau-\sigma)F(|\eta|,\sigma)\mathrm{sgn}[(\xi-x-\eta)\eta]d\eta d\sigma$$

is a solution of the boundary value problem

$$u_t = u_{xx} + F(x,t)$$

$$u(0,t) = 0, \quad \Phi_\xi\{u(\xi,t)\} = 0$$
$$\chi_\tau\{u(x,\tau)\} = 0,$$

provided $F(x,t)$ satisfies the condition

$$\chi_\tau\{F(x,\tau)\} = 0.$$

The proof could be carried out by checking directly the above representations. Such a proof however does not reveal how these representations are obtained. Here we propose a sketch of the method by which they are derived.

First we consider the right inverse operator L of $\dfrac{d^2}{dt^2}$ in the space $C[0,a]$ of the continuous functions in $[0,a]$, defined as the solution of the elementary boundary value problem

$$y'' = f(x)$$
$$y(0) = 0, \quad \Phi\{y\} = 0.$$

The operator $y = Lf(x)$ has the form

$$Lf(x) = \int_0^x (x-\xi)f(\xi)d\xi - \frac{x}{\Phi\{\xi\}}\Phi\left\{\int_0^\xi (\xi-\eta)f(\eta)d\eta\right\},$$

provided $\Phi\{\xi\} \neq 0$. For simplicity we assume $\Phi\{\xi\} = 1$. Then

$$Lf(x) = \int_0^x (x-\xi)f(\xi)d\xi - x\Phi\left\{\int_0^\xi (\xi-\eta)f(\eta)d\eta\right\}. \tag{5}$$

According to [3] the operation

$$(f \overset{t}{*} g)(x) = -\frac{1}{2}\Phi\left\{\int_0^\xi h(x,\eta)d\eta\right\},$$

where

$$\begin{aligned} h(x,\xi) \;=\; & \left[\int_x^\xi f(\xi+x-\eta)g(\eta)d\eta\right. \\ & \left. - \int_{-x}^\xi f(|\xi-x-\eta|)g(|\eta|)\operatorname{sgn}[(\xi-x-\eta)\eta]d\eta\right] \end{aligned} \tag{6}$$

is a convolution of L in $C[0,a]$, such that

$$Lf(x) = \{x\} \overset{x}{*} f(x).$$

Next we consider the right inverse operator l of $\dfrac{d}{dt}$ in the space $C[0,\infty)$ defined as the solution of the elementary boundary value problem

$$z' = \varphi(t), \quad \chi\{z\} = 0.$$

In order this problem to have a solution it is necessary and sufficient $\chi\{1\} \neq 0$. Again we may assume for simplicity that $\chi\{1\} = 1$. Then $z = l\varphi(t)$ has the form

$$l\varphi(t) = \int_0^t \varphi(\tau)d\tau - \chi\{\int_0^\tau \varphi(\sigma)d\sigma\}.$$

According to [3] the operation

$$(\varphi \overset{t}{*} \psi)(t) = \chi_\tau \left\{ \int_\tau^t \varphi(t+\tau-\sigma)\psi(\sigma)d\sigma \right\}$$

is a convolution of l in $C[0,\infty)$ such that

$$l\varphi = \{1\} \overset{t}{*} \varphi.$$

Further we consider the space $C(\Delta)$ of the continuous functions in the half-strip $\Delta = [0,a]\times[0,\infty)$ and the operators L and l in $C(\Delta)$, acting "partialy" with respect to the variables x and t, correspondingly, i.e.

$$L\{u(x,t)\} = \int_0^x (x-\xi)u(\xi,t)d\xi - x\Phi_\xi \left\{ \int_0^\xi (\xi-\eta)u(\eta,t)d\eta \right\} \tag{7}$$

and

$$l\{u(x,t)\} = \int_0^t u(x,\tau)d\tau - \chi_\tau \left\{ \int_0^\tau u(x,\sigma)d\sigma \right\}. \tag{8}$$

We are looking for a convolution $u * v$ for L and l in $C(\Delta)$, such that

$$(Ll)u = \{x\} * u.$$

Lemma. *Let $u, v \in C(\Delta)$. The operation*

$$(u * v)(x,t) = -\frac{1}{2}\Phi_\xi \circ \chi_\tau \left\{ \int_0^\xi h(x,t;\eta,\tau)d\eta \right\},$$

where

$$h(x,t;\eta,\tau) = \int_x^\xi \int_\tau^t u(\xi+x-\eta, t+\tau-\sigma)v(\eta,\sigma)d\eta d\sigma$$

$$-\int_{-x}^{\xi}\int_{\tau}^{t}\tilde{u}(\xi-x-\eta,t+\tau-\sigma)\tilde{v}(\eta,\sigma)d\eta d\sigma$$

with $\tilde{u}(x,t)=u(|x|,t)\operatorname{sgn}x$, $\tilde{v}(x,t)=v(|x|,t)\operatorname{sgn}x$, *is a bilinear, commutative and associative operation in* $C(\Delta)$, *such that* L *and* l *are multipliers of the algebra* $(C(\Delta),*)$ *and the representation*

$$(Ll)u=\{x\}*u$$

holds.

The proof may be accomplished in a manner completely analogous to the proof of the corresponding assertion in [5].

In order to prove that (4) is a solution of the boundary value problem considered it remains to use some elementary properties of the convolution $u*v$, since

$$u(x,t)=\frac{\partial}{\partial t}(\Omega*F)$$

Remarks

1) The time-nonlocal initial value conditions are not so often considered. Nevertheless, we can make at least two references to such initial value conditions treated by other authors (see Dezin [2], Lattes et Lions [1]).

In [2] the linear functional

$$\chi(\varphi)=\mu\varphi(T)-\mu(0)$$

with $\mu\neq 0,1$ is considered.

In [1] the boundary functional

$$\chi(\varphi)=\int_{-T}^{T}\varphi(\tau)d\tau$$

is considered and the condition (1) is called a "thick" initial value condition.

2) The operation

$$(\varphi*\psi)(t)=\chi_\tau\{\int_{\tau}^{t}\varphi(t+\tau-\sigma)\psi(\sigma)d\sigma$$

is a convolution generalizing the Duhamel convolution

$$(\varphi*\psi)(t)=\int_{0}^{t}\varphi(t-\tau)\psi(\tau)d\tau.$$

It is found by the second author. Its properties (see [3]) are studied thoroughly in [4].

References

[1] R. Lattès, J.-L. Lions. *The Method of Quasy-Reversibility. Application to PDE*, Elsevier, N. York, 1974.

[2] A. A. Dezin. *Partial Differential Equations. An Introduction to a General Theory of Linear Boundary Value Problems*, Springer Verlag, Berlin - Heidelberg - New York, 1987.

[3] I. H. Dimovski. *Convolutional Calculus*, Kluwer, Dordrecht - N. York - London, 1990.

[4] N. S. Bozhinov. *Convolutional Representations of Commutants and Multipliers*, Publishing House of Bulgarian Academy of Sciences, Sofia, 1988.

[5] G. I. Chobanov, I. H. Dimovski. A two-variate operational calculus for boundary value problems, *Fractional Calculus and Applied Analysis* **2**, No 5, 1999, 591-601.

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