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Duhamel Representations for a Class of Non-Local Boundary Value Problems [1]

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The solutions of many common linear boundary value problems for the partial differential equations of mathematical physics have Duhamel type representations of the form $D(\Omega \star f)$, where D is a differential operator, Ω is a special solution of the same problem, $\star$ is a non-classical convolution and f is a given function.

Our aim is to show that in many cases essential simplifications of these representations are possible in order they to allow computer implementation.

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1. Introduction

In [1], using operational calculus approach, Duhamel type representations of boundary value problems for the heat and the wave equations are obtained.

For the heat equation

$$u_t = u_{xx} + F(x,t)$$

in the half-strip $\Delta = [0,a] \times [0,\infty)$ it is considered the boundary value problem

$$u(0,t) = 0, \quad u(a,t) = 0, \quad u(x,0) = f(x).$$

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The following representation is obtained for its solution:

$$u(x,t) = \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}(\Omega \overset{(x)}{\star} f) + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}(\Omega \overset{(x,t)}{\star} F), \tag{1}$$

where $\overset{(x)}{\star}$ and $\overset{(x,t)}{\star}$ are non-classical convolutions and Ω is the solution of the same problem for the special choice $f(x) = 0$, $F(x,t) = x$, having the explicit form

$$\Omega(x,t) = \frac{x}{6}(a^2 - x^2) + 2\frac{a^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{a}\exp\left(-\left(\frac{n\pi}{a}\right)^2 t\right).$$

An analogous representation is obtained for the solution of the wave equation

$$v_{tt} = v_{xx} + F(x,t)$$

with the boundary value conditions

$$v(0,t) = 0, \;\; u(a,t) = 0, \;\; v(x,0) = f(x), \;\; v_t(x,0) = g(x).$$

For the heat equation (with $a = \pi$) a simpler representation is known (see Widder [2], Kartashov [3]):

$$u(x,t) = \int_0^{\pi}[\theta(x-\xi,t) - \theta(x+\xi,t)]f(\xi)d\xi + \int_0^{\pi}\int_0^{t}[\theta(x-\eta,t) - \theta(x+\eta,t)]F(\eta,t)d\tau d\eta,$$

where $\theta(x,t) = (1/2\pi) + (1/\pi)\sum\limits_{n=1}^{\infty}\exp(-n^2 t)\cos nx$ is the well-known Jacobi θ-function.

Our aim here is two-fold:

1) To generalize the boundary value problems considered in [1] in order to encompass also a class of non-local boundary value problems, and to obtain Duhamel type representations of the solutions;

2) To simplify these representations for some boundary value functionals.

2. Two classes of boundary value problems (local and non-local) for the heat and wave equations

Let Φ be a continuous non-zero linear functional in $C^1[0,1]$. Then we consider the following two classes of boundary value problems in the half-strip $\Delta = [0,a] \times [0,\infty)$:

A) $u_t = u_{xx} + F(x,t)$
$u(0,t) = \Phi_\xi\{u(\xi,t)\} = 0$
$u(x,0) = f(x)$

B) $v_{tt} = v_{xx} + F(x,t)$
$v(0,t) = \Phi_\xi\{v(\xi,t)\} = 0$
$v(x,0) = f(x), \ \ v_t(x,0) = g(x).$

In order to obtain explicit Duhamel-type representations of the solutions of the boundary value problem considered, we use an operational calculus approach.

3. A general operational calculus

I) Operational calculus for the operator $D = d^2/dx^2$ **with the boundary value conditions** $y(0) = 0, \Phi\{y\} = 0$. Here we restrict our considerations to the case when $\lambda = 0$ is not an eigenvalue of the eigenvalue problem

$$y'' + \lambda^2 y = 0, \ y(0) = 0, \ \Phi\{y\} = 0 \tag{2}$$

This is equivalent to the requirement $\Phi\{x\} \neq 0$. Without loss of generality we may assume $\Phi\{x\} = 1$.

Then we introduce the right inverse operator L of $D = d^2/dx^2$ in $C[0,1]$, defined as the solution ($Lf(x) = y$) of the simplest boundary value problem

$$y'' = f(x), \ y(0) = 0, \ \Phi\{y\} = 0.$$

The explicit expression for $Lf(x)$ can be easily found:

$$Lf(x) = \int_0^x (x-\xi)f(\xi)d\xi - x\Phi_\xi\left\{\int_0^\xi (\xi-\eta)f(\eta)d\eta\right\}. \tag{3}$$

In [4] the following convolution for L is proposed:

$$(f \overset{(x)}{*} g)(x) = -\frac{1}{2}\Phi_\xi \circ l_\xi\{h(x,\xi)\}, \tag{4}$$

where

$$h(x,\xi) = \int_x^\xi f(\xi + x - \eta)g(\eta)d\eta - \int_{-x}^\xi f_1(\xi - x - \eta)g_1(\eta)d\eta,$$

with $f_1(x) = f(|x|)\mathrm{sgn}x, \ g_1(x) = g(|x|)\mathrm{sgn}x$. L is the convolution operator $\{x\}\overset{(x)}{*}$, i.e. L: $Lf = \{x\}\overset{(x)}{*} f$.

The construction of the corresponding operational calculus for L goes in the lines of Section 3 of [1]. For the details see [4]. Using the approach by the multiplier quotients, proposed in [4], we define the algebraic inverse of L as the fraction

$$S = \frac{1}{L}. \tag{5}$$

Here by 1 is denoted the identity operator.

The basic formula of the corresponding operational calculus is:

$$f'' = Sf - (1 - \Phi\{1\}x)f(0)S - \Phi\{f\}, \tag{6}$$

where $\Phi\{f\}$ is a "numerical operator" in the sense of Mikusinski [5].

Formula (6) is equivalent to the identity

$$Lf'' = f - \{1 - \Phi\{1\}x\}f(0) - x\Phi\{f\}. \tag{7}$$

It can easily be obtained by direct check.

For the next considerations we are to obtain some differentiation properties of convolution (3).

Lemma 1. *If $\Phi \in (C[0,a])^\star$ and $f, g \in C[0,a]$, then the function $(f \overset{(x)}{\star} g)(x)$ is continuously differentiable, and*

$$\frac{d}{dx}\left(f \overset{(x)}{\star} g\right) = -\frac{1}{2}\Phi_\xi\{k(x,\xi)\}, \tag{8}$$

where $k(x,\xi) = \int\limits_x^\xi f(\xi + x - \eta)g(\eta)d\eta + \int\limits_{-x}^\xi f_1(\xi - x - \eta)g_1(\eta)d\eta.$

P r o o f. First, we prove (8) for $f, g \in C^1[0,a]$:

$$\frac{d}{dx}\left(f \overset{(x)}{\star} g\right) = -\frac{1}{2}\frac{d}{dx}\Phi_\xi \circ \int\limits_0^\xi h(x,\eta)d\eta = -\frac{1}{2}\Phi_\xi \circ \int\limits_0^\xi \frac{\partial h(x,\eta)}{\partial x}d\eta$$

$$= -\frac{1}{2}\Phi_\xi \circ \int\limits_0^\xi \frac{\partial k(x,\eta)}{\partial \eta}d\eta = -\frac{1}{2}\Phi_\xi\{k(x,\xi) - k(x,0)\} = -\frac{1}{2}\Phi_\xi\{k(x,\xi)\}.$$

Further, using the density of $C^1[0,a]$ in $C[0,a]$, and the closed graph theorem, the identity (8) follows for arbitrary $f, g \in C[0,a]$. ■

Lemma 2. *Let $\Phi = \Psi \circ l$, where $\Psi \in (C[0,a])^\star$. If $f, g \in C[0,a]$, then $(f \overset{(x)}{\star} g)(x)$ is twice continuously differentiable, i.e. $(f \star g) \in C^2[0,a]$, and*

$$\frac{d^2}{dx^2}(f \overset{(x)}{\star} g) = f(x)\Phi(g) + g(x)\Phi(f) - \Psi\{1\}\int\limits_0^x f(x-\eta)g(\eta)d\eta - \frac{1}{2}\Psi_\xi\{h(x,\xi). \tag{9}$$

First, (9) should be proved for $f, g \in C^1[0,a]$ and then (9) to be extended by continuity on $C[0,a]$. Here we omit the details.

Remark. The operation

$$(f\tilde{\star}g) = \frac{d^2}{dx^2}(f \overset{(x)}{\star} g)$$

$$= f(x)\Phi(g) + g(x)\Phi(f) - \Psi\{1\}\int\limits_0^x f(x-\eta)g(\eta)d\eta - \frac{1}{2}\Psi_\xi\{h(x,\xi)\}$$

is a convolution of the operator L in $C[0,a]$ such that the function $\{x\}$ is the unit of the convolution algebra $(C[0,a], \tilde{\star})$, i.e. $\{x\}\tilde{\star}f = f$ for each $f \in C[0,a]$.

As for the operator L in (10), it can be represented by the function

$$L\{x\} = \left\{x^3/6 - (x/6)\Phi\{\xi^3\}\right\}, \text{i.e.} Lf = \left\{(x^3/6) - (x/6)\Phi\{\xi^3\}\right\}\tilde{\star}f.$$

II. A two-dimensional operational calculus. Let us consider the space $C(\Delta)$, $\Delta = [0,a] \times [0,\infty)$ and the operators

$$Lu(x,t) = \int\limits_0^x (x-\eta)u(\eta,t)d\eta - x\Phi_\xi\left\{\int\limits_0^\xi (\xi-\eta)u(\xi,\eta)d\eta\right\} = \{x\} \overset{(x)}{\star} \{u(x,t)\},$$

$$lu(x,t) = \int\limits_0^t u(x,\tau)d\tau = \{1\} \overset{(t)}{\star} \{u(x,t)\},$$

where by $\overset{(t)}{\star}$ the classical Duhamel convolution $(\varphi \overset{(t)}{\star} \psi)(t) = \int\limits_0^t \varphi(t-\tau)\psi(\tau)d\tau$ is denoted. Our aim here is to construct an operational calculus on $C(\Delta)$ for both operators l and L.

Theorem 1. *Let $\tilde{\Phi}$ be the linear functional $\Phi_\xi \circ \int\limits_0^\xi$. Then the operation*

$$(F \star G)(x,t) = -\frac{1}{2}\tilde{\Phi}_\xi\left\{\int\limits_0^t h(x,t;\xi,\tau)d\tau\right\}, \tag{10}$$

with

$$h(x,t;\xi,\tau) = \int\limits_x^\xi F(\xi + x - \eta, t - \tau)G(\eta,\tau)d\eta - \int\limits_{-x}^\xi F_1(\xi - x - \eta, t-\tau)G_1(\eta,\tau)d\eta,$$

$F_1(x,t) = F(|x|,t)\mathrm{sgn}x$, $G_1(x,t) = G(|x|,t)\mathrm{sgn}x$, *is a convolution of* l *and* L *in* $C(\Delta)$ *such that* $LlF = \{x\} \star F$.

The proof could be accomplished in the lines of the corresponding proof in [1].

Remark. The operators l and L are multipliers of the convolution algebra $(C(\Delta),\star)$, i.e. $l(F\star G) = (lF)\star G$ and $L(F\star G) = (LF)\star G$.

For the next consideration we shall use the representations $L\{F(x,t)\} = \{x\} \overset{(x)}{\star} F$ and $l\{F(x,t)\} = \{1\} \overset{(t)}{\star} F$, where $\overset{(x)}{\star}$ denotes the operation (4) and $\overset{(t)}{\star}$ denotes the usual Duhamel convolution.

The operational calculus which can be developed using convolution (10) will be used for the solution of the boundary value problems **A** and **B**.

Following the multiplier quotients approach, then along with the operators L and l, basic elements of the corresponding operational calculus are their algebraic inverses $s = 1/l$, $S = 1/L$.

Definition 1. Let $\phi(t) \in C[0,\infty)$. Then the convolution operator $\left(\phi \overset{(t)}{\star}\right) u(x,t) = \int\limits_0^t \phi(t-\tau)u(x,\tau)d\tau$ is said to be a numerical operator with respect to x. Denotation: $[\phi(t)]_x$.

Definition 2. Let $f(x) \in C[0,a]$. Then the convolution operator $\left(f \overset{(x)}{\star}\right) u(x,t) = f \overset{(x)}{\star} \{u(x,t)\}$ is said to be a numerical operator with respect to t. Denotation: $[f(x)]_t$.

Lemma 3. *If* $u \in C(\Delta)$ *is twice continuously differentiable with respect to* x *and once with respect to* t, *then* $\partial^2 u/\partial x^2$ *and* $\partial u/\partial t$ *are given by*

$$\frac{\partial^2 u}{\partial x^2} = Su - \{u(0,t)(1-\Phi\{1\}x)\}f(0)S - [\Phi_\xi\{u(\xi,t)\}]_x \tag{11}$$

and

$$\frac{\partial u}{\partial t} = su - [u(x,0)]_t, \tag{12}$$

where by subscripts x and t numerical operators with respect to x and t are denoted.

4. Applications to some non local boundary value problem

The use of the operational calculus approach for the solution of the boundary value problem considered goes in three steps:

1. algebraizing of the problem;
2. algebraic solution of the problem;
3. interpretation of the algebraic solution as a function solution.

Further we consider both boundary value problems **A** and **B** in parallel. Using formulae (11) and (12), we obtain

$$(s - S)u = [f(x)]_t + \{F(x,t)\} \tag{13}$$

and

$$(s^2 - S)v = s[f(x)]_t + [g(x)]_t + \{F(x,t)\}. \tag{14}$$

It is important to know whether $s - S$ and $s^2 - S$ are divisors of 0 or not. The answer is given by the following

Theorem 2. *If $a \in \operatorname{supp} \Phi$, then $s - S$ and $s^2 - S$ are non-divisors of 0 in the multiplier quotients ring of $[C(\Delta), \star]$.*

P r o o f. We prove only the assertion for $s - S$ since the proof for $s^2 - S$ is similar. Assume the converse, i.e. that

$$(s - S)A = 0 \tag{15}$$

for a nonzero multiplier quotient A. Since in the convolution algebra $(C(\Delta), \star)$ there are non-divisor of zero, then A can be represented in the form

$$A = \frac{u(x,t)\star}{v(x,t)\star},$$

where $u = \{u(x,t)\}$ and $v = \{v(x,t)\}$ are elements of $(C(\Delta), \star)$ such that v is non-divisor of 0 of the convolution algebra $(C(\Delta), \star)$.

Since $s = 1/l$ and $S = 1/L$, then (15) is equivalent to the identity

$$(L - l)u = 0 \tag{16}$$

in $C(\Delta)$. We are to show that (16) implies $u = 0$.

Further, we use the generalized Fourier transform (see Dimovski and Petrova [6])

$$f(x) \mapsto P_n\{f\} = \{\varphi_n(x)\} \overset{(x)}{*} \{f(x)\}, \quad n \in \mathbf{N}, \tag{17}$$

where $\varphi_n(x) = (1/2\pi) \int\limits_{\Gamma n} L_{-\lambda^2} f(x) \lambda d\lambda$ with a simple contour Γ_n around λ_n.

Let κ_n be the multiplicity of the eigenvalue $-\lambda_n^2$. Then we denote

$$\varphi_n^{(0)}(x) = \varphi_n(x),$$

$$\varphi_n^{(k)}(x) = \left(L + \frac{1}{\lambda_n^2}\right)^k \varphi_n(x), \quad 0 \le k \le \kappa_n - 1.$$

Applying (17) to (16) we obtain

$$(L - l)u_n = 0, \qquad (u_n(x,t) = \{\varphi_n(x)\} \overset{(x)}{*} \{u(x,t)\}). \tag{18}$$

Since the functions $\varphi_n^{(k)}$, $0 \le k \le \kappa_n - 1$ form a basis of the space $\ker(L + 1/\lambda_n^2)^{\kappa_n}$, then $Lu_n(x,t) = \sum\limits_{k=0}^{\kappa_n - 1} A_k(t) L\varphi_n^{(k)}(x)$, $n = 1, 2, \ldots$ with unknown continuous functions $A_k(t)$.

Then

$$Lu_n(x,t) = \sum_{k=0}^{\kappa_n - 1} A_k(t) L\varphi_n^{(k)}(x) = \sum_{k=0}^{\kappa_n - 1} A_k(t) \left(-\frac{1}{\lambda_n^2}\varphi_n^{(k)}(x) + \varphi_n^{(k+1)}(x)\right),$$

where $\varphi_n^{(\kappa_n)} = 0$. Thus we obtain the following chain of equations

$$\frac{1}{\lambda_n^2} A_0(t) + l A_0(t) = 0$$

$$\frac{1}{\lambda_n^2} A_1(t) + l A_1(t) - A_0(t) = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{\lambda_n^2} A_{\kappa_n - 1}(t) + l A_{\kappa_n - 1}(t) - A_{\kappa_n - 2}(t) = 0.$$

We obtain consequently: $A_0(t) = 0, A_1(t) = 0, A_{\kappa_n - 1}(t) = 0$ and hence, $u_n(x,t) \equiv 0$. From the assumption $a \in \operatorname{supp}\Phi$ it follows the totality of the projectors (17) (see Dimovski and Petrova [6], Theorem 5). Therefore $u(x,t) = 0$ for every fixed t, $0 \le t < \infty$. ■

Remark. The assertion of Theorem 1 is equivalent to the uniqueness of the solutions of the boundary value problems **A** and **B**, provided $a \in \operatorname{supp}\Phi$.

When the uniqueness is proven, then the second step is the easiest one. From (13) and (14) we get

$$u = \frac{1}{s - S}[f(x)]_t + \frac{1}{s - S}\{F(x,t)\}, \tag{19}$$

$$v = \frac{s}{s^2 - S}[f(x)]_t + \frac{1}{s^2 - S}[g(x)]_t + \frac{1}{s^2 - S}\{F(x,t)\}. \tag{20}$$

In order to interpret the right-hand sides of (19) and (20) as ordinary functions, we proceed as follows. Denoting

$$U = \frac{1}{S(s - S)}, \quad V = \frac{1}{S(s^2 - S)}$$

and assuming that U and V are functions from $C(\Delta)$, we can write

$$\frac{1}{s - S}[f(x)]_t = S\left(\frac{1}{S(s - S)}[f(x)]_t\right) = \frac{\partial^2}{\partial x^2}\left(U \overset{(x)}{\ast} f(x)\right)$$

and

$$\frac{1}{s^2 - S}[g(x)]_t = \frac{\partial^2}{\partial x^2}\left(V \overset{(x)}{\ast} g(x)\right).$$

Theorem 3. *If the problem **A** with $f(x) \equiv x$ and $F(x,t) \equiv 0$ has a function solution $U(x,t)$, then*

$$u(x,t) = \frac{\partial^2}{\partial x^2}\left(U \overset{(x)}{\ast} f(x)\right) \tag{21}$$

*is a function solution of **A** with general ("arbitrary") f.*

Remark. It is clear that in order **A** to have a function solution $u(x,t) \in C(\Delta)$, it is necessary to assume $f(0) = 0$ and $\Phi\{f\} = 0$. Further "mild" restrictions on $f(x)$ may be necessary.

Theorem 3$'$. *If the problem **B** with $f(x) \equiv 0$, $g(x) \equiv x$ and $F(x,t) \equiv 0$ has a function solution $V(x,t)$, then*

$$v(x,t) = \frac{\partial^2}{\partial x^2}\left(V \overset{(x)}{\ast} g(x)\right) \tag{22}$$

*is a function solution of **B** with "arbitrary" g.*

5. Simplification of Duhamel type representations (21) and (22)

Case 1. Let $\Phi \in (C[0,a])^\star$, $f(x)$ and $U(x,t)$ be functions as in Theorem 3;

$$k(x,\xi) = \int_x^\xi U(\xi + x - \eta, t) f(\eta) d\eta + \int_{-x}^\xi U_1(\xi - x - \eta, t) f_1(\eta) d\eta,$$

$f_1(x) = f(|x|)\mathrm{sgn}x$, $U_1(x,t) = U(|x|,t)\mathrm{sgn}x$.

Example 1. Consider the problem **A** in the special case $a = 1$, $\Phi\{f\} = f(1)$, $F(x,t) = 0$. From Lemma 1 it follows that $u(x,t) = \frac{\partial}{\partial x}\left(-\frac{1}{2}\Phi_\xi\{k(x,\xi)\}\right)$. Since (see [1]) $U(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) e^{-(n\pi)^2 t}$, then $u(x,t) = -\frac{1}{2}\frac{\partial}{\partial x} k(x,1)$. Further t for the solution $u(x,t)$ we get

$$u(x,t) = \int_0^1 \left[\theta(x-\eta,t) - \theta(x+\eta,t)\right] f(\eta) d\eta, \tag{23}$$

where $\theta(x,t)$ is the Jacobi θ-function $\theta(x,t) = 1/2 + \sum_{n=1}^{\infty} \cos(n\pi x) e^{-(n\pi)^2 t}$.

Case 2. Let $\Phi = \Psi \circ l$, $\Psi \in (C[0,a])^\star$, $g(x)$ and $V(x,t)$ be functions as in Theorem 3′,

$$h(x,\xi) = \int_x^\xi V(\xi + x - \eta, t) g(\eta) d\eta - \int_{-x}^\xi V_1(\xi - x - \eta, t) g_1(\eta) d\eta,$$

$g_1(x) = g(|x|)\mathrm{sgn}x$, $V_1(x,t) = V(|x|,t)\mathrm{sgn}x$.

Example 2. Consider problem **B** in the special case $a = 1$, $\Phi\{\varphi\} = 2\int_0^1 \varphi(\xi) d\xi$, $f(x) = 0$, arbitrary $g(x)$ and $F(x,t) = 0$. From Lemma 2 it follows

$$v(x,t) = 2g(x)\Phi_\xi\{V(\xi,t)\} + 2V(x,t)\Phi\{g\} - 2\Psi\{1\}\int_0^x V(x-\eta,t) g(\eta) d\eta - \Psi_\xi\{h(x,\xi)\},$$

where

$$V(x,t) = \sum_{n=1}^{\infty}\left[\frac{-x\cos(2n\pi x)\sin(2n\pi t)}{n\pi} + \frac{\sin(2n\pi t) - 2n\pi t\cos(2n\pi t)}{2(n\pi)^2}\sin(2n\pi x)\right].$$

Since the above series is slowly convergent, it is not suitable for computer realization. That is why we consider the problem with $f(x) = 0$ and $g(x) = x^3/6 - x/12$.

If $\Omega(x,t)$ is the solution of this problem, then

$$v(x,t) = \frac{\partial^4}{\partial x^4}(\Omega \star g), \tag{24}$$

where

$$\Omega(x,t) = -\sum_{n=1}^{\infty}\left[\frac{\sin(2n\pi t)}{(2n\pi)^3}(-2x\cos(2n\pi x)) + \left(\frac{6}{(2n\pi)^4}\sin(2n\pi t) - \frac{2}{(2n\pi)^3}t\cos(2n\pi t)\right)\sin(2n\pi x)\right].$$

Representation (24) is rather complicated for a numerical calculation, since it contains a derivative of 4-th order. Happily enough, it can be simplified up to

$$v(x,t) = -\Omega_x(0,t)g(x) - 2\int_0^x \Omega_x(x-\eta,t)g'(\eta)d\eta - \int_x^1 \Omega_x(1+x-\eta,t)g'(\eta)d\eta - \int_{-x}^1 \Omega_x(1-x-\eta,t)g'(|\eta|)d\eta, \tag{25}$$

where

$$\Omega_x(x,t) = -2\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^2}\left[\frac{1}{n\pi}\sin(2n\pi t)\cos(2n\pi x) + x\sin(2n\pi t)\sin(2n\pi x) - t\cos(2n\pi t)\cos(2n\pi x))\right];$$

$$\Omega_x(0,t) = -\frac{1}{12}(4\{t^3\} - \{t\}) + \frac{t}{2}(\{t^2\} - \{t\} + 1/6);$$

we use the denotation {t} for the fractional part of t, i.e. $\{t\} = t - [t]$. Representation (25) can be used successfully for a numerical calculation of the solution, provided we assume implicitly that $g(x)$ is continuously differentiable with a possible exception of finite number of points.

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