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Computer Implementation of Solutions of BVP for Finite Vibrating Systems [1]

Ivan H. Dimovski *, *Margarita N. Spiridonova* **

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The study of linear elastic vibrating systems (strings, beams, plates, shells) leads to various boundary value problems (BVP) for equations of 2^{nd} and higher orders. As a basic tool, usually, the Fourier method is used. However, the computer implementation of this method faces difficulties connected with the slow convergence of the series involved. The authors propose a way for avoiding these difficulties. On representative examples it is shown how extensions of the Duhamel principle to the space variables enables to obtain a closed form solution of various problems for finite vibrating systems. To this end, as in the classical Duhamel principle, one special solution of the same problem, but for a very simple and special choice of the initial value functions should be obtained, using, say, the Fourier method. Then the solution for arbitrary initial value functions can be obtained in the form of a non-classical convolution. The use of such Duhamel-type representations for numerical solution of a number of vibrating problems by means of the computer algebra system *Mathematica* is illustrated.

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1. Introduction

One of the first examples of application of the Fourier method is the problem of vibrating of a finite string:

$$\begin{aligned} &u_{tt} = u_{xx} + F(x,t), \quad 0 < x < a, \; 0 < t < \infty, \\ &u(0,t) = 0, \; u(a,t) = 0 \\ &u(x,0) = f(x), \; u_t(x,0) = g(x) \end{aligned}$$

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For the homogeneous equation ($F \equiv 0$) the solution has the form:

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left(a_n \sin \frac{n\pi t}{a} + b_n \cos \frac{n\pi t}{a} \right), \tag{1}$$

where

$$a_n = \frac{2}{n\pi} \int_0^a g(x) \sin \frac{n\pi x}{a} dx, \quad b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx, \quad n = 1,2,3,\ldots$$

are the coefficients of the Fourier sine expansion of the initial value functions.

From the computational point of view, a shortcoming of the explicit solution (1) is the necessity to calculate many Fourier coefficients of the initial value functions and to sum the series (1) in all points, where the values of the solution are needed. These two operations are rather time-consuming. That is why the Fourier method hardly could be used, say, in real-time control problems.

In our approach these two steps ((1) of expansion of the initial value functions $f(x)$ and $g(x)$ into Fourier sine series, and (2) of summing the series (1) in all points, where the solution is needed) are avoided. For the sake of simplicity, let us consider the case when $f(x) \equiv 0$ and $g(x)$ be "arbitrary" chosen.

In order to write down an explicit Duhamel type representation of the solution $u(x,t)$, one should find the solution for the special choice $f(x) \equiv 0$, $g(x) \equiv x$, i.e.

$$\Omega(x,t) = \frac{2a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \frac{n\pi x}{a} \sin \frac{n\pi t}{a},$$

and then to write down the general solution in the form

$$u(x,t) = \frac{\partial^2}{\partial x^2} \left\{ \Omega(x,t) \overset{(x)}{*} g(x) \right\}, \tag{2}$$

where the convolution operation $\overset{(x)}{*}$ has the form

$$\varphi(x) \overset{(x)}{*} \psi(x) = -\frac{1}{2a} \int_0^a h(x,\xi) d\xi, \tag{3}$$

with

$$h(x,t) = \int_x^{\xi} \varphi(\xi + x - \eta)\psi(\eta)d\eta - \int_{-x}^{\xi} \varphi(|\xi - x - \eta|)\psi(|\eta|)\mathrm{sgn}\left[(\xi - x - \eta)\eta\right] d\eta.$$

In such a form the solution for $f(x) \equiv 0$ and arbitrary $g(x)$ is obtained in Dimovski and Chobanov [2].

2. Simplifying the basic Duhamel representation

Further, the idea is to use the Duhamel-type representation (2) for a computer implementation. But representation (2) looks rather involved. In fact, it can be simplified. Indeed, the convolution (3) allows a differentiation in an explicit form

$$(4) \qquad \frac{d}{dx}\left[\varphi(x) \overset{(x)}{*} \psi(x)\right] = -\frac{1}{2a}\left\{\int_x^a \varphi(a+x-\xi)\psi(\xi)d\xi \right.$$
$$\left. + \int_{-x}^a \varphi(|a-x-\xi|)\psi(|\xi|)\operatorname{sgn}\left[(a-x-\xi)\xi\right] d\xi\right\}.$$

Example 1. Consider the equation of a vibrating string

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \ t > 0$$

with the boundary and initial value conditions

$$u(0,t) = 0, \ u(1,t) = 0$$
$$u(x,0) = 0, \ u_t(x,0) = g(x).$$

Assuming that $g(x)$ is differentiable (or consists of finite number of such pieces) and satisfies the conditions $g(0) = g(1) = 0$, representation (2) takes the form

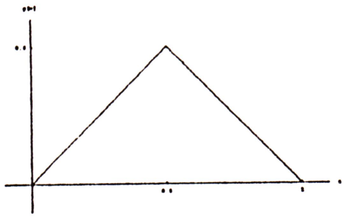
$$(5) \qquad u(x,t) = -\frac{1}{2}\int_x^1 \Omega(1+x-\xi)g'(\xi)d\xi$$
$$+\frac{1}{2}\int_{-x}^1 \Omega(1-x-\xi)g'(|\xi|)d\xi,$$

where

$$\Omega(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^{n-1}/n^2)\sin n\pi x \sin n\pi t$$

is the solution of the same problem but for the special choice $g(x) \equiv x$.

The above representation is convenient for computation of the solution in arbitrary point of the domain. Experimental computations are made in the environment of the computer algebra system *Mathematica* [6]. On Figure 1 visualization of the computed solution is shown, preceded by the graph of the chosen function $g(x)$.



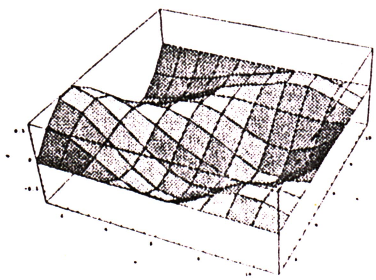


Figure 1

E x a m p l e 2. Consider the equation of a free supported beam (see Farlow [5]):

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4},\ 0 < x < 1,\ 0 < t < \infty,$$

with the initial-boundary value conditions:

$$u(0,t) = 0,\ u_{xx}(0,t) = 0,\ u(1,t) = 0,\ u_{xx}(1,t) = 0$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

Using the series solution in Farlow [5], we have

(6) $$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin(n\pi)^2 t + b_n \cos(n\pi)^2 t\right] \sin n\pi x\,,$$

where

$$a_n = \frac{2}{(n\pi)^2} \int_0^1 g(x) \sin(n\pi x) dx,\ b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx,\ n = 1, 2, 3, \ldots$$

Further, we consider the case when $f(x) \equiv 0$. Then from representation (5), we obtain

(7) $$u(x,t) = -\frac{1}{2} \int_x^1 \Omega_x(1 + x - \xi, t) g(\xi) d\xi$$

$$+\frac{1}{2}\int_{-x}^{1}\Omega_x(1-x-\xi,t)g(|\xi|)\mathrm{sgn}\xi d\xi\,,$$

where

$$\Omega_x(x,t)=\frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^{n-1}/n^2)\sin(n\pi)^2 t\cos n\pi x\,.$$

On Figure 2 the relief of the solution is shown, together with the graph of the chosen function $g(x)$. All computations are made using the system *Mathematica* again.

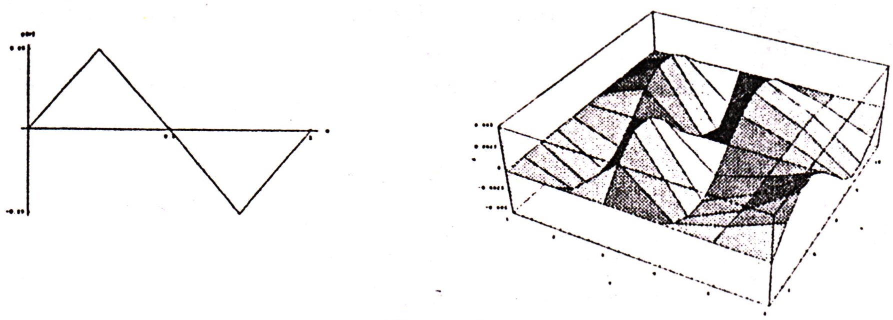


Figure 2

3. A non-local boundary value problem

The approach by Duhamel-type representation with respect to the space variables works for non-local BVP too.

A famous non-local BVP is the so-called Samarskii–Ionkin problem for the heat equation:

$$u_{tt}=u_{xx},$$
$$u(0,t)=0,\ \int_0^1 u(\xi,t)d\xi=0.$$
$$u(x,0)=f(x)$$

In Dimovski ([1], [3], [4]) the explicit convolution

$$(f*g)(x)=2f(x)\int_0^1 g(\xi)d\xi+2g(x)\int_0^1 f(\xi)d\xi$$

$$-2\int_0^x f(x-\xi)g(\xi)d\xi-\int_x^1 f(1+x-\xi)g(\xi)d\xi$$

$$+\int_{-x}^1 f(|1+x-\xi|,t)g(|\xi|)\mathrm{sgn}\left[(\xi(1-x-\xi)\right]d\xi$$

is used to obtain the representation

$$u(x,t) = \Omega(x,t) \overset{(x)}{*} f(x),$$

where

$$\Omega(x,t) = \sum_{n=1}^{\infty} \{-2x \cos 2n\pi x + 8\pi n t \sin 2n\pi x\} e^{-4n^2\pi^2 t}$$

is the solution of the same problem, but for the special choice $f(x) \equiv x$.

A visualization of the solution in the case when $f(x)$ has the same form as in Example 2 is shown in [4].

For an illustrating example now we consider a new type of boundary value problem.

Example 3. Consider the following BVP:

(8)
$$\begin{array}{l} u_{tt} = u_{xx}, \quad 0 < x < 1,\ 0 < t < \infty \\ u(0,t) = 0, \ \int_0^1 u(\xi,t)d\xi = 0 \\ u(x,0) = 0, \ u_t(x,0) = g(x)\,. \end{array}$$

Here again

(9)
$$u(x,t) = \tilde{\Omega}(x,t) \overset{(x)}{*} g(x),$$

where $\tilde{\Omega}(x,t)$ is the solution of the same problem but for $g(x) \equiv x$.

Formally, we obtain

(10)
$$\Omega(x,t)$$
$$= \sum_{n=1}^{\infty} \left\{ \left(\frac{\sin 2\,n\,\pi\,x}{2n^2\pi^2} - \frac{x \cos 2\,n\,\pi\,x}{n\pi} \right) \sin 2n\pi t - \frac{t \sin 2\,n\,\pi\,x}{n\pi} \cos 2\,n\,\pi\,t \right\}.$$

Unfortunately, the series is very slow convergent and it hardly could be used for numerical calculation of the solution using formula (9).

Instead of the solution of (8) for $f(x) \equiv x$, we consider the solution $\Omega(x,t)$ for $f(x) \equiv x^3/6 - x/12$. The function $f(x)$ satisfies both the boundary value conditions $f(0) = 0$ and $\int_0^1 f(\xi)d\xi = 0$.

For $\Omega(x,t)$ we obtain

$$\Omega(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{x\, \cos(2\,n\,\pi\,x)\, \sin(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} \right.$$
$$\left. + \left(\frac{t\, \cos(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} - \frac{3\, \sin(2\,n\,\pi\,t)}{8\,n^4\,\pi^4} \right) \sin(2\,n\,\pi\,x) \right\}.$$

Then the solution $u(x,t)$ has the form

$$u(x,t) = \frac{\partial^2}{\partial x^2}\left\{\Omega(x,t) \overset{(x)}{*} f(x)\right\}. \tag{11}$$

It is a matter of simple check to see that

$$u(x,t) = -2\Omega_x(0,t)g(x) - 2\int_0^x \Omega_x(x-\xi,t)g'(\xi)d\xi$$
$$-\int_x^1 \Omega_x(1+x-\xi,t)g'(\xi)d\xi - \int_{-x}^1 \Omega_x(1-x-\xi,t)g'(|\xi|)d\xi,$$

where

$$\Omega_x(x,t) = \sum_{n=1}^{\infty}\left\{\frac{\cos(2\,n\,\pi\,x)\,\sin(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} + 2\,n\,\pi\,\cos(2\,n\,\pi\,x)\right. \tag{12}$$
$$\left.\times\left(\frac{t\,\cos(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} - \frac{3\,\sin(2\,n\,\pi\,t)}{8\,n^4\,\pi^4}\right) - \frac{x\,\sin(2\,n\,\pi\,t)\,\sin(2\,n\,\pi\,x)}{2\,n^2\,\pi^2}\right\}.$$

This representation is rather convenient for computer implementation. At the experimental computations, made with *Mathematica* system, different values for the place n of truncating of the series (12) were used. On Figure 3 the relief of the numerical solution for $n = 20$ is shown. In this case $g(x)$ has the same form as in Example 2.

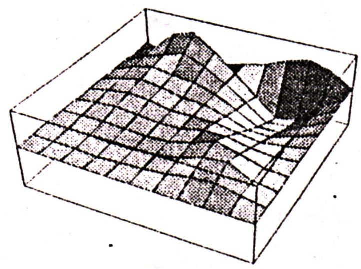


Figure 3

4. Concluding remarks

We would like to point out some advantages of the use of Duhamel-type representations of the solutions of the considered BVP, in comparison with some known numerical methods.

- Compared with the Fourier method, the two time-consuming operations mentioned above are avoided. In such a way, the proposed approach has all the advantages of the Fourier method and avoids most of its shortcomings.
- Comparing with the use of difference methods, the values of the solution can be obtained for the points where they are needed only and the stability problem does not occur.

The computations related to the presented approach are performed by means of the computer algebra system *Mathematica*. The programs for these computations are part of a microenvironment under development, allowing investigation and application of the considered approach.

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*,** *Institute of Mathematics and Informatics*
Bulgarian Academy of Sciences
Acad. G. Bontchev Str., Block 8
Sofia 1113, BULGARIA

e-mail: mspirid@math.bas.bg

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