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Two Dimensional Spectral Problems Containing Delta Distribution or Conjugation Conditions

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In this paper non standard spectral problems for Laplace operator in a square domain with various boundary conditions and conjugation conditions on the line interface are considered. It is proved that the eigenfunctions can be expressed by means of sine-type functions while the eigenvalues satisfy some transcendental equation. The spectral problems of this type are of a great importance in investigation of solutions of the initial boundary value problems for parabolic and hyperbolic equations with concentrated factors.

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1. Heat conduction problem with concentrated capacity

Let $\Omega = \{x|\ x = (x_1, x_2),\ 0 < x_i < 1,\ i = 1,2\}$, $Q = \Omega \times (0,T)$, $\Gamma = \partial\Omega \times (0,T)$ and S be the segment $\{(\xi, x_2)|\ 0 \leq x_2 \leq 1\}$, $\xi \in (0,1)$. We consider the parabolic problem

$$(1)\qquad [1 + K\delta_S]\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad (x,t) \in Q,$$

$$(2)\qquad u|_\Gamma = 0,$$

$$(3)\qquad u(x,0) = u_0(x), \quad x \in \Omega.$$

Here $\delta_S = \delta_S(x)$ is Dirac's distribution concentrated on S and $K = K(x_2) \in L^\infty(0,1)$, $0 < K_1 \leq K \leq K_2$. Equation (1) models heat conduction process with concentrated capacity on interface S (see [1], [3], [4]).

It can be easily verified that the solution of (1)-(3) satisfies the equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad x \in \Omega^- \cup \Omega^+, \tag{4}$$

$$\Omega^- = (0,\xi) \times (0,1),\ \Omega^+ = (\xi,1) \times (0,1),$$

initial and boundary conditions (3), (2) and conditions of conjugation:

$$[u]_S = u(\xi+0,x_2,t) - u(\xi-0,x_2,t) = 0, \quad \left[\frac{\partial u}{\partial n}\right]_S = K\frac{\partial u}{\partial t}\Big|_S, \tag{5}$$

where $\frac{\partial}{\partial n} = \frac{\partial}{\partial x_1}$ is the normal derivative with respect to the external normal to S.

2. Abstract setting of the problem

Let H be a Hilbert space endowed with an inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. For a linear, selfadjoint, unbounded, positive definite operator A with domain $D(A)$ dense in H, we define in a usual way the energy space H_A with inner product $(u,v)_A = (Au,v)$ and norm $\|\cdot\|_A$. Then problem (1)–(3) can be written as an abstract Cauchy problem (see [5])

$$B\frac{dU}{dt} + AU = 0, \quad t > 0; \quad U(0) = u_0,$$

where B is a linear selfadjoint unbounded positive definite operator with domain $D(B) \subset H$ and A is unbounded in H_B. In our case $H = L^2(\Omega)$, $Au = -\Delta u$ and $Bu = [1 + K\delta_S]u$. Then $H_A = \overset{\circ}{W}{}_2^1(\Omega)$ and

$$\|w\|_A^2 = \int_\Omega \left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2\right] dx; \quad \|w\|_B^2 = \int_\Omega w^2(x)\,dx + \int_S K w^2 dx_1.$$

3. Energy estimate

In order to obtain an energy estimate for the solution of the problem (1)–(3), we take the product of (4) with $u(x,t)$ and integrate the result on Ω:

$$\iint_\Omega u\frac{\partial u}{\partial t} dx_1\,dx_2 = \iint_\Omega u\,\Delta u\,dx_1\,dx_2.$$

Using a variant of Green's formula on the right hand side of the equality, we obtain:

$$\iint_\Omega u\,\Delta u\,dx_1\,dx_2 = \iint_{\Omega^-\cup\Omega^+} u\,\Delta u\,dx_1\,dx_2$$
$$= -\iint_{\Omega^-\cup\Omega^+}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]dx_1\,dx_2 - \int_S u\left[\frac{\partial u}{\partial x_1}\right]_S dx_2.$$

On the other hand,

$$\iint_\Omega u\frac{\partial u}{\partial t}\,dx_1\,dx_2 = \frac{1}{2}\frac{d}{dt}\iint_\Omega u^2\,dx_1\,dx_2.$$

By the second condition of conjugation, we get

$$\int_S u\left[\frac{\partial u}{\partial x_1}\right]_S dx_2 = \int_S Ku\frac{\partial u}{\partial t}dx_2 = \frac{1}{2}\frac{d}{dt}\int_S Ku^2\,dx_2.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\left[\iint_\Omega u^2\,dx_1\,dx_2 + \int_S Ku^2\,dx_2\right] = -\iint_\Omega\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]dx_1\,dx_2,$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\|u\|_B^2 = -\|u\|_A^2.$$

We define λ_1 by

$$\frac{1}{\lambda_1} = \sup_{w\in H_A}\frac{\|w\|_B^2}{\|w\|_A^2}.$$

It can be shown (see [2]) that λ_1 is the first positive eigenvalue of the spectral problem

$$Aw = \lambda\,B\,w\,,$$

which has a discrete set of eigenvalues, while the eigenfunctions satisfy the condition of orthogonality and represent a basis of the space H_B.

For the considered model problem, the spectral problem reads as follows

$$\text{(6)}\qquad \begin{aligned} &-\quad \Delta w = \lambda w, \quad x\in\Omega\backslash S,\\ &w(x)|_{\partial\Omega} = 0,\\ &[w]_S = w(\xi+0,x_2)-w(\xi-0,x_2)=0, \quad -\left[\frac{\partial w}{\partial x_1}\right]_S = \lambda K w|_S. \end{aligned}$$

Taking into account the inequality $\|w\|_A^2 \geq \lambda_1 \|w\|_B^2$, $w \in H_A$, we get

$$\frac{d}{dt}\|u\|_B^2 = -2\|u\|_A^2 \leq -2\lambda_1 \|u\|_B^2.$$

Integrating this inequality, and using the initial condition, the following estimate can be obtained:

$$\|u\|_B^2 \leq \|u_0\|_B^2 \, e^{-2\lambda_1 t},$$

or

$$\iint_\Omega u^2(x,t)\,dx_1\,dx_2 + \int_S K u^2(x,t)\,dx_2$$
$$\leq e^{-2\lambda_1 t}\left[\iint_\Omega u_0^2(x)\,dx_1\,dx_2 + \int_S K u_0^2(x)\,dx_2\right].$$

4. Dirichlet's spectral problem

In the sequel, we will assume that K is a constant. In this case the solution of the spectral problem (6) can be written in the form $w(x_1,x_2) = v(x_1)y(x_2)$, where

$$v(x_1) = \begin{cases} A\sin\alpha x_1, & x_1 \in (0,\xi) \\ B\sin\alpha(1-x_1), & x_1 \in (\xi,1) \end{cases}, \quad y(x_2) = \sin j\pi x_2,\ j = 1,2,\ldots$$

It is obvious that $w(x_1,x_2)$ satisfies the boundary conditions. The values of the constants A and B can be obtained from the first condition of conjugation: $A = \sin\alpha(1-\xi)$, $B = \sin\alpha\xi$. The equation $-\Delta w = \lambda w$ gives $\lambda = \alpha^2 + j^2\pi^2$. Using the second condition of conjugation, we obtain:

$$\frac{1}{K}[\cot\alpha(1-\xi) + \cot\alpha\xi] = \frac{\alpha^2 + j^2\pi^2}{\alpha}. \tag{7}$$

In some cases, there exists another family of eigenfunctions vanishing on the interface. Consequently, normal derivatives of these eigenfunctions are continuous on S. Such eigenfunctions exist only if ξ is rational, i.e. $\xi = \frac{p}{q}$. Then, $\alpha = n\,q\,\pi$, the eigenvalues are $\lambda_{nj} = \alpha_n^2 + (j\pi)^2$ and corresponding eigenfunctions are

$$w_{nj} = v_n(x_1)y_j(x_2), \qquad \begin{array}{ll} v_n(x_1) = \sin nq\pi x_1, & n = 1,2,\ldots \\ y_j(x_2) = \sin j\pi x_2, & j = 1,2,\ldots \end{array}$$

If $\xi = 0.5$, the equation (7) takes the form

$$\frac{2}{K}\cot\frac{\alpha}{2} = \frac{\alpha^2 + j^2\pi^2}{\alpha}. \tag{8}$$

For each value of $j \in N$ the equation has a countable set of solutions $\alpha_i, i \in N$. The graphical solutions of the equation (8) for $j = 1, 2, 3, 4$ is shown in **Figure 1**, and the numerical values of α_{ij}, $i, j = 1, 2, 3$ are shown in **Table 1**.

$i \backslash j$	1	2	3
1	6.7605	6.5985	6.4807
2	12.8579	12.8167	12.7685
3	19.0532	19.0385	19.0180
4	25.2882	25.2815	25.2715

Table 1

In **Figure 2**, the eigenfunctions w_{ij} for $i = 1, 2, 3$ and $j = 1, 2$ are presented.

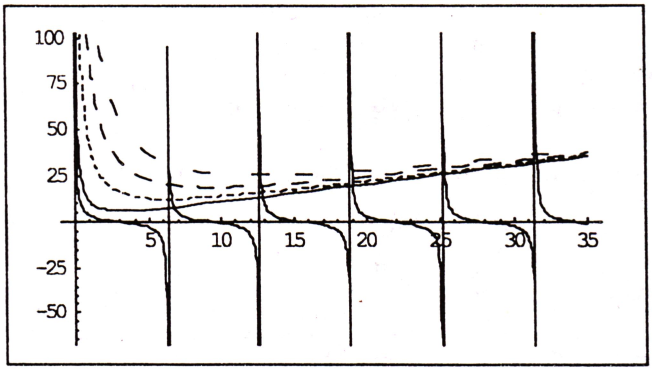

Figure 1

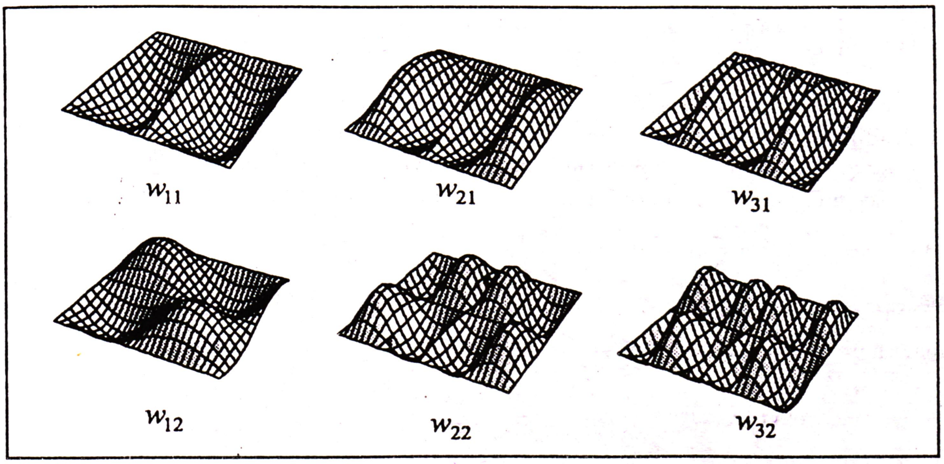

Figure 2

5. Neumann's spectral problem

Let us consider the heat equation with concentrated capacity (1) with initial value (3) and Neumann's boundary conditions

$$\frac{\partial u}{\partial n}\Big|_{\Gamma} = 0. \tag{9}$$

The problem (1), (3), (9) can be written in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad x \in \Omega^- \cup \Omega^+,$$

$$\frac{\partial u}{\partial n}\Big|_{\Gamma} = 0; \quad u(x,0) = u_0(x),\ x \in \Omega,$$

$$[u]_S = u(\xi+0, x_2, t) - u(\xi-0, x_2, t) = 0, \quad \left[\frac{\partial u}{\partial n}\right]_S = K\frac{\partial u}{\partial t}\Big|_S.$$

Using the same procedure as above, one obtains the following spectral problem:

$$-\ \Delta w = \lambda w, \quad x \in \Omega\backslash S,$$

$$\frac{\partial w}{\partial n}\Big|_{\partial\Omega} = 0,$$

$$[w]_S = 0, \quad -\left[\frac{\partial w}{\partial x_1}\right]_S = \lambda K w|_S.$$

The solution of this problem can be written in the form $w(x_1,x_2) = v(x_1)y(x_2)$, where

$$v(x) = \begin{cases} A\cos\alpha x_1, & x_1 \in (0,\xi) \\ B\cos\alpha(1-x_1), & x_1 \in (\xi,1) \end{cases}, \quad y(x_2) = \cos j\pi x_2,\ j = 1,2,\ldots$$

It is obvious that $w(x_1,x_2)$ satisfies the boundary conditions. Using the first condition of conjugation, the values of the constants are $A = \cos\alpha(1-\xi)$, $B = \cos\alpha\xi$. Taking into account the equality $-\Delta w = \lambda w$, we get $\lambda = \alpha^2 + j^2\pi^2$. By the second condition of conjugation the following equality can be obtained:

$$-\frac{1}{K}[\tan\alpha(1-\xi) + \tan\alpha\xi] = \frac{\alpha^2 + j^2\pi^2}{\alpha}. \tag{10}$$

Parasite solutions occur, if $\xi = \dfrac{2k+1}{2m}$, $k = 0,1,2\ldots, m = 1,2,\ldots$. Then, there exist eigenfunctions

$$w_{nj} = v_n(x_1)y_j(x_2), \quad \begin{array}{ll} v_n(x_1) = \cos m\,(2n+1)\pi\, x_1, & n = 0,1,2,\ldots \\ y_j(x_2) = \cos j\pi x_2, & j = 1,2,\ldots \end{array}$$

while the corresponding eigenvalues are $\lambda_{nj} = [m(2n+1)\pi]^2 + (j\pi)^2$.

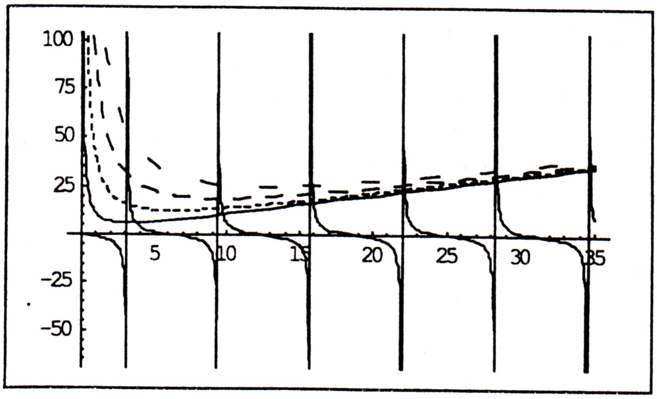

Figure 3

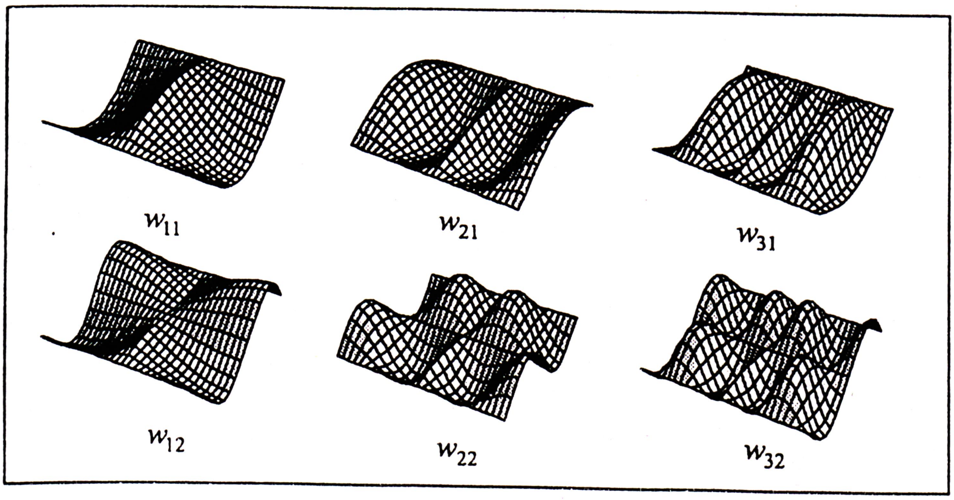

Figure 4

If $\xi = 0.5$, the equation (10) takes the form

$$-\frac{2}{K}\tan\frac{\alpha}{2} = \frac{\alpha^2 + j^2\pi^2}{\alpha}.$$

For each value of $j \in N$ there exist a countable set of solutions $\alpha_i, i \in N$ of the equation. The graphical solutions of this equation is shown in **Figure 3**, while the numerical solutions for α_{ij}, $i, j = 1, 2, 3$ are presented in **Table 2**.

$i\backslash j$	1	2	3
1	3.7490	3.4068	3.2729
2	9.7910	9.7131	9.6361
3	15.9482	15.9245	15.8936
4	22.1676	22.1578	22.1483

Table 2

In **Figure 4**, the eigenfunctions w_{ij} are presented.

Analogous results are obtained for the spectral problem with third boundary condition.

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