Mathematica Balkanica

New Series Vol. 18, 2004, Fasc. 3-4

Polynomial as a General Solution of a Linear Differential Equation of Fourth Order

Elena J. Hadzieva [1], *Ilija A. Shapkarev* [2]

Presented at Internat. Congress "MASSEE' 2003", 4th Symposium "TMSF"

In this article we consider a homogeneous linear differential equation of fourth order with polynomial coefficients. The degree of each coefficient is the same as the order of the derivative which is multiplied by. Sufficient conditions for existence of general polynomial solution are obtained. Such a solution is constructed, depending on the zeros of the polynomial of the highest degree.

AMS Subj. Classification: 34-A05

Key Words: differential equations, general solution, polynomial solution, particular solution, existence conditions, construction of the solution

A linear differential equation of fourth order

$$a(x)y^{IV} + b(x)y''' + c(x)y'' + d(x)y' + e(x)y = 0 \tag{1}$$

is considered, where

$$a(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4, \quad b(x) = B_0 + B_1x + B_2x^2 + B_3x^3,$$
$$c(x) = C_0 + C_1x + C_2x^2, \quad d(x) = D_0 + D_1x, \quad e(x) = E_0,$$

and A_i, B_i, C_i, D_i, E_i (i takes allowable values) are constants, $A_4 \neq 0$. By differentiating the above equation n times (n is natural), we get:

$$\begin{aligned} & a(x)y^{(n+4)} + \left[\binom{n}{1}a'(x) + \binom{n}{0}b(x)\right]y^{(n+3)} \\ & + \left[\binom{n}{2}a''(x) + \binom{n}{1}b'(x) + \binom{n}{0}c(x)\right]y^{(n+2)} \\ & + \left[\binom{n}{3}a'''(x) + \binom{n}{2}b''(x) + \binom{n}{1}c'(x) + \binom{n}{0}d(x)\right]y^{(n+1)} \end{aligned} \tag{2}$$

$$+ \left[\binom{n}{4}a^{IV}(x) + \binom{n}{3}b'''(x) + \binom{n}{2}c''(x) + \binom{n}{1}d'(x) + \binom{n}{0}e(x)\right]y^{(n)} = 0.$$

Further, we consider the case

$$\binom{n}{4}a^{IV}(x)+\binom{n}{3}b'''(x)+\binom{n}{2}c''(x)+\binom{n}{1}d'(x)+\binom{n}{0}e(x)=0, \tag{3}$$

$$\binom{n}{3}a'''(x)+\binom{n}{2}b''(x)+\binom{n}{1}c'(x)+\binom{n}{0}d(x)\neq 0, \tag{4}$$

because other cases are considered in [3]. As it is known, [3,4], the equation (3) is necessary and sufficient condition for the equation (1) to have polynomial solution of n-th degree. Now, in view of (3), differential equation (1) becomes:

$$\begin{aligned} &a(x)y^{(n+4)}+\left[\binom{n}{1}a'(x)+\binom{n}{0}b(x)\right]y^{(n+3)}\\ &+\left[\binom{n}{2}a''(x)+\binom{n}{1}b'(x)+\binom{n}{0}c(x)\right]y^{(n+2)}\\ &+\left[\binom{n}{3}a'''(x)+\binom{n}{2}b''(x)+\binom{n}{1}c'(x)+\binom{n}{0}d(x)\right]y^{(n+1)}=0. \end{aligned} \tag{5}$$

Since

$$\begin{aligned} &\binom{n}{3}a'''(x)+\binom{n}{2}b''(x)+\binom{n}{1}c'(x)+\binom{n}{0}d(x)\\ &=(4n(n-1)(n-2)A_4+3n(n-1)B_3+2nC_2+D_1)x\\ &\quad+n(n-1)(n-2)A_3+n(n-1)B_2+nC_1+D_0. \end{aligned}$$

In order to abbreviate the next expressions, we denote:

$$\alpha=4n(n-1)(n-2)A_4+3n(n-1)B_3+2nC_2+D_1,$$

$$\beta=n(n-1)(n-2)A_3+n(n-1)B_2+nC_1+D_0.$$

Putting

$$a(x)=\frac{1}{p}(Ax^3+Bx^2+Cx+D)(\alpha x+\beta), \tag{6}$$

$$\binom{n}{1}a'(x)+\binom{n}{0}b(x)=\frac{1}{p}(Ex^2+Fx+G)(\alpha x+\beta), \tag{7}$$

$$\binom{n}{2}a''(x)+\binom{n}{1}b'(x)+\binom{n}{0}c(x)=\frac{1}{p}(Hx+I)(\alpha x+\beta), \tag{8}$$

where A, B, C, D, E, F, G, H, I are constants yet to be specified and p is a non-zero constant, the equation (1) takes the following form:

(9)
$$(Ax^3+Bx^2+Cx+D)y^{(n+4)}+(Ex^2+Fx+G)y^{(n+3)}+(Hx+J)y^{(n+2)}+py^{(n+1)}=0.$$

By differentiating the last equation $(m-1)$-times, m being a natural, we get:

$$\begin{aligned} &\left(Ax^3+Bx^2+Cx+D\right)y^{(n+m+3)} \\ &+\left[(3(m-1)A+E)\,x^2+(2(m-1)B+F)\,x+(m-1)C+G\right]y^{(n+m+2)} \\ &+\left[(3(m-1)(m-2)A+2(m-1)E+H)\,x\right. \\ &\left.+(m-1)(m-2)B+(m-1)F+I\right]y^{(n+m+1)} \\ &+\left[(m-1)(m-2)(m-3)A+(m-1)(m-2)E+(m-1)H+p\right]y^{(n+m)}=0. \end{aligned} \tag{10}$$

Because of (10), a sufficient condition for (1) to have a polynomial solution of degree $(n+m)$ is [1,2,5]:

$$(m-1)(m-2)(m-3)A+(m-1)(m-2)E+(m-1)H+p=0 \tag{11}$$

which changes (10) into the form:

$$\begin{aligned} &\left(Ax^3+Bx^2+Cx+D\right)y^{(n+m+3)} \\ &+\left[(3(m-1)A+E)\,x^2+(2(m-1)B+F)\,x+(m-1)C+G\right]y^{(n+m+2)} \\ &+\left[(3(m-1)(m-2)A+2(m-1)E+H)\,x\right. \\ &\left.+n(m-1)(m-2)B+(m-1)F+I\right]y^{(n+m+1)}=0. \end{aligned} \tag{12}$$

Provided that

(13)
$$(3(m-1)(m-2)A+2(m-1)+H)\,x+(m-1)(m-2)B+(m-1)F+I\neq 0,$$

if we denote

$$\gamma=3(m-1)(m-2)A+2(m-1)+H,$$
$$\delta=(m-1)(m-2)B+(m-1)F+I$$

and put

$$Ax^3+Bx^2+Cx+D=\frac{1}{q}(Lx^2+Mx+N)(\gamma x+\delta) \tag{14}$$

(15)
$$(3(m-1)A+E)\,x^2+(2(m-1)B+F)\,x+(m-1)C+G=\frac{1}{q}(Jx+K)(\gamma x+\delta),$$

where L, M, N, J, K are constants yet to be specified and q is a non-zero constant, then the equation (12) reduces to

$$(Lx^2+Mx+N)y^{(n+m+3)}+(Jx+K)y^{(n+m+2)}+qy^{(n+m+1)}=0. \tag{16}$$

If we differentiate the last equation $(k-1)$- times (k being a natural), we obtain

$$\begin{aligned}&(Lx^2+Mx+N)y^{(n+m+k+2)}+[2(k-1)L+J)x+(k-1)M+K]\\&\times y^{(n+m+k+1)}+[(k-1)(k-2)L+(k-1)J+q]\,y^{(n+m+k)}=0.\end{aligned} \tag{17}$$

Because of (17), a sufficient condition for (1) to have a polynomial solution of degree $(n+m+k)$ is

$$(k-1)(k-2)L+(k-1)J+q=0, \tag{18}$$

which transforms the previous equation into the form
(19)
$$(Lx^2+Mx+N)y^{(n+m+k+2)}+[2(k-1)L+J)x+(k-1)M+K]\,y^{(n+m+k+1)}=0.$$

If we suppose that

$$(2(k-1)L+J)x+(k-1)M+K\neq 0, \tag{20}$$

denote

$$\zeta=2(k-1)L+J,\quad \eta=(k-1)M+K,$$

and put

$$Lx^2+Mx+N=\frac{1}{s}(Px+Q)(\zeta x+\eta), \tag{21}$$

(P and Q will be determined later, $s\neq 0$), then the equation (19) takes the form:

$$(Px+Q)y^{(n+m+k+2)}+sy^{(n+m+k+1)}=0. \tag{22}$$

We differentiate (22) $(j-1)$-times and get

$$(Px+q)y^{(n+m+k+j+1)}+((j-1)P+s)y^{(n+m+k+j)}=0. \tag{23}$$

In order the equation (1) to have a polynomial solution of degree $(n+m+k+j)$, the following condition is sufficient:

$$(j-1)P+s=0. \tag{24}$$

Now, let us determine the unknown constants A, B, C, D, E, F, G, H, I, L, M, N, J, K. By equating the coefficients associated to each degree of x in the equations (6),(7),(8),(14),(15),(21), we get the following:

$$A=\frac{p}{\alpha}A_4, \qquad B=\frac{p}{\alpha}\left(A_3-\frac{\beta}{\alpha}A_4\right),$$
$$C=\frac{p}{\alpha}\left(A_2-\frac{\beta}{\alpha}A_3-\frac{\beta^2}{\alpha^2}A_4\right), \qquad D=\frac{p}{\alpha}\left(A_1-\frac{\beta}{\alpha}A_2+\frac{\beta^2}{\alpha^2}A_3-\frac{\beta^3}{\alpha^3}A_4\right),$$

$$A_0-\frac{\beta}{\alpha}A_1+\frac{\beta^2}{\alpha^2}A_2-\frac{\beta^3}{\alpha^3}A_3+\frac{\beta^4}{\alpha^4}A_4=0, \tag{25}$$

$$E=\frac{p}{\alpha}(4nA_4+B_3), \qquad F=\frac{p}{\alpha}\left(3nA_3+B_2-\frac{\beta}{\alpha}(4nA_4+B_3)\right),$$
$$G=\frac{p}{\alpha}\left(2nA_2+B_1-\frac{\beta}{\alpha}(3nA_3+B_2)+\frac{\beta^2}{\alpha^2}(4nA_4+B_3)\right),$$

$$nA_1+B_0-\frac{\beta}{\alpha}(2nA_2+B_1)+\frac{\beta^2}{\alpha^2}(3nA_3+B_2)-\frac{\beta^3}{\alpha^3}(4nA_4+B_3), \tag{26}$$

$$\begin{aligned} H &= \frac{p}{\alpha}(6n(n-1)A_4+3nB_3+C_2),\\ I &= \frac{p}{\alpha}\left(3n(n-1)A_3+2nB_2+C_1-\frac{\beta}{\alpha}(6n(n-1)A_4+3nB_3+C_2)\right), \end{aligned}$$

$$\begin{aligned} &n(n-1)A_2+nB_1+C_0-\frac{\beta}{\alpha}(3n(n-1)A_3+2nB_2+C_1)\\ &+\frac{\beta^2}{\alpha^2}(6n(n-1)A_4+3nB_3+C_2)=0, \end{aligned} \tag{27}$$

$$L=\frac{qp}{\gamma\alpha}A_4, \quad M=\frac{qp}{\gamma\alpha}\left[A_3-\left(\frac{\beta}{\alpha}+\frac{\delta}{\gamma}\right)A_4\right],$$
$$N=\frac{qp}{\gamma\alpha}\left[A_2-\left(\frac{\beta}{\alpha}+\frac{\delta}{\gamma}\right)A_3+\left(\frac{\delta^2}{\gamma^2}+\frac{\delta}{\gamma}\cdot\frac{\beta}{\alpha}+\frac{\beta^2}{\alpha^2}\right)A_4\right],$$

$$
\begin{aligned}
(28)\qquad & A_1-\left(\frac{\beta}{\alpha}+\frac{\delta}{\gamma}\right)A_2+\left(\frac{\beta^2}{\alpha^2}+\frac{\beta}{\alpha}\cdot\frac{\delta}{\gamma}+\frac{\delta^2}{\gamma^2}\right)A_3\\
& -\left(\frac{\beta^3}{\alpha^3}+\frac{\beta^2}{\alpha^2}\frac{\delta}{\gamma}+\frac{\beta}{\alpha}\frac{\delta^2}{\gamma^2}+\frac{\delta^3}{\gamma^3}\right)A_4=0,
\end{aligned}
$$

$$
\begin{aligned}
J &= \frac{qp}{\gamma\alpha}\left[(4n+3m-3)A_4+B_3\right],\\
K &= \frac{qp}{\gamma\alpha}\left[(3n+2m-2)A_3+B_2-\frac{\beta}{\alpha}((4n+2m-2)A_4+B_3)\right.\\
&- \left.\frac{\delta}{\gamma}((4n+3m-3)A_4+B_3\right],
\end{aligned}
$$

$$
\begin{aligned}
(29)\qquad & (2n+m-1)A_2+B_1-\frac{\beta}{\alpha}((3n+m-1)+B_2)\\
& +\frac{\beta^2}{\alpha^2}((4n+m-1)A_4+B_3)-\frac{\delta}{\gamma}((3n+2m-2)A_3+B_2)\\
& +\frac{\beta}{\alpha}\frac{\delta}{\gamma}((4n+3m-2)A_4+B_3)+\frac{\delta^2}{\gamma^2}((4n+3m-3)A_4+B_3)=0,
\end{aligned}
$$

$$
P=\frac{spq}{\alpha\gamma\zeta}A_4,\quad Q=\frac{spq}{\alpha\gamma\zeta}\left(A_3-\left(\frac{\beta}{\alpha}+\frac{\delta}{\gamma}+\frac{\eta}{\zeta}\right)\right),
$$

$$
(30)\quad A_2-\left(\frac{\beta}{\alpha}+\frac{\delta}{\gamma}+\frac{\eta}{\zeta}\right)A_3+\left(\frac{\beta^2}{\alpha^2}+\frac{\delta^2}{\gamma^2}+\frac{\eta^2}{\zeta^2}+\frac{\beta}{\alpha}\frac{\delta}{\gamma}+\frac{\beta}{\alpha}\frac{\eta}{\zeta}+\frac{\delta}{\gamma}\frac{\eta}{\zeta}\right)A_4=0.
$$

Remark. $x_1=-\dfrac{\beta}{\alpha}$ is a root of the equation $a(x)=0$, as it can be seen from (25). The other 3 roots are $x_2=-\dfrac{\delta}{\gamma}$, $x_3=-\dfrac{\eta}{\zeta}$, $x_4=-\dfrac{Q}{P}$ (see (14) and (21)).

So far the unknown coefficients are determined in terms of A_i, B_i, C_i, D_i, E_i (i takes allowable values) and x_1, x_2, x_3. Further calculations give interesting results. From (24), by expressing P and ζ, we get

$$
B_3=-(4n+3m+2k+j-6)A_4.
$$

The equation (18), including the expressions for L and J gives us:

$$
C_2 = [(n+m+k+j-2)(n+m+k-2)+
$$

$$+(n+m+k+j-2)(n+m-1)+(n+m+k-1)(n+m-1)$$
$$+(n+m+k+j-2)n+(n+m+k-1)n+(n+m)n]\,A_4.$$

Similarly, we can find the expressions for D_1, using (11) and E_0, using (3)

$$\begin{aligned} D_1 &= -[(n+m+k+j-1)(n+m+k-1)(n+m-1) \\ &\quad +(n+m+k+j-1)(n+m+k-1)n \\ &\quad +(n+m+k+j-1)(n+m)n+(n+m+k)(n+m)n]\,A_4, \\ E_0 &= (n+m+k+j)(n+m+k)(n+m)n. \end{aligned}$$

For the next considerations, the Viet rules will be useful, applied to the equation $a(x)=0$. From the fact that $x_3=-\dfrac{\eta}{\zeta}$, using the expressions for ζ and η and further for L, M, J, K, and using one of the Viet rules, we get that

$$\begin{aligned} B_2 &= [(3n+3m+2k+j-6)x_1+(3n+2m+2k+j-5)x_2+ \\ &\quad +(3n+2m+k+j-4)x_3+(3n+2m+k-3)x_4]\,A_4. \end{aligned}$$

The condition (29) and the previous results give us the expression for B_1,

$$\begin{aligned} B_1 &= -[(2n+2m+2k+j-5)x_1x_2+(2n+2m+k+j-4)x_1x_3 \\ &\quad +(2n+2m+k-3)x_1x_4+(2n+m+k+j-3)x_2x_3 \\ &\quad +(2n+m+k-2)x_2x_4+(2n+m-1)x_3x_4]\,A_4. \end{aligned}$$

Further, we get

$$\begin{aligned} B_0 &= [(n+m+k+j-3)x_1x_2x_3+(n+m+k-2)x_1x_2x_4 \\ &\quad +(n+m-1)x_1x_3x_4+nx_2x_3x_4]\,A_4 \end{aligned}$$

from (26), and

$$\begin{aligned} C_1 &= -[(n+m+k+j-2)(n+m+k-2)x_1 \\ &\quad +(n+m+k+j-2)(n+m-1)x_1+(n+m+k-1)(n+m-1)x_1 \\ &\quad +(n+m+k+j-2)(n+m+k-2)x_2+(n+m+k+j-2)nx_2 \\ &\quad +(n+m+k-1)nx_2+(n+m+k+j-2)(n+m-1)x_3 \\ &\quad +(n+m+k+j-2)nx_3+(n+m)nx_3+(n+m)nx_4 \\ &\quad +(n+m+k-1)(n+m-1)x_4+(n+m+k-1)nx_4]\,A_4, \end{aligned}$$

knowing that $x_2=-\dfrac{\delta}{\gamma}$. From the condition (27), C_0 can be obtained,

$$C_0 = [(n+m+k+j-2)(n+m+k-2)x_1x_2$$

$$+(n+m+k+j-2)(n+m-1)x_1x_3$$
$$+(n+m+k-1)(n+m-1)x_1x_4+(n+m+k+j-2)nx_2x_3$$
$$+(n+m+k-1)nx_2x_4+(n+m)nx_3x_4]\,A_4,$$

and from the fact that $x_1=-\dfrac{\beta}{\alpha}$, for D_0 we obtain:

$$\begin{aligned} D_0 \;=\; & [(n+m+k+j-1)(n+m+k-1)(n+m-1)x_1 \\ & +(n+m+k+j-1)(n+m+k-1)nx_2 \\ & +(n+m+k+j-1)(n+m)nx_3+(n+m+k)(n+m)nx_4]\,A_4. \end{aligned}$$

Taking into account what we obtained for $B_0, B_1, B_2, B_3, C_0, C_1, C_2, D_0, D_1, E_0$, the equation (1) transforms into:

$$\begin{aligned} & (x-x_1)(x-x_2)(x-x_3)(x-x_4)y^{IV} \\ & -[(n+m+k+j-3)(x-x_1)(x-x_2)(x-x_3) \\ & +(n+m+k-2)(x-x_1)(x-x_2)(x-x_4) \\ & +(n+m-1)(x-x_1)(x-x_3)(x-x_4)+n(x-x_2)(x-x_3)(x-x_4)]\,y''' \\ & +[(n+m+k+j-2)(n+m+k-2)(x-x_1)(x-x_2) \\ & +(n+m+k+j-2)(n+m-1)(x-x_1)(x-x_3) \\ & +(n+m+k-1)(n+m-1)(x-x_1)(x-x_4) \\ & +(n+m+k+j-2)n(x-x_2)(x-x_3) \\ & +(n+m+k-1)n(x-x_2)(x-x_4)+(n+m)n(x-x_3)(x-x_4)]\,y'' \\ & -[(n+m+k+j-1)(n+m+k-1)(n+m-1)(x-x_1)+ \\ & +(n+m+k+j-1)(n+m+k-1)n(x-x_2) \\ & +(n+m+k+j-1)(n+m)n(x-x_3) \\ & +(n+m+k)(n+m)n(x-x_4)]\,y' \\ & +(n+m+k+j)(n+m+k)(n+m)ny=0. \end{aligned}$$

The last equation can be reduced to the following system:

$$\begin{aligned} (x-x_4)y'-(n+m+k+j)y &= z \\ (x-x_3)z'-(n+m+k)z &= u \\ (x-x_2)u'-(n+m)u &= v \\ (x-x_1)v'-nv &= 0 \end{aligned}$$

which can be easily solved. The solution in view of y is

$$y=K_1(x-x_4)^{n+m+k+j}$$

$$+K_2 \sum_{i=0}^{n+m+k} \binom{n+m+k}{i} \frac{(x_4-x_3)^i}{i+j} (x-x_4)^{n+m+k-i}$$
$$+K_3 \sum_{i=0}^{n+m} \binom{n+m}{i} \frac{(x_3-x_2)^i}{k+i} \sum_{s=0}^{n+m-i} \binom{n+m-i}{s} \frac{(x_4-x_3)^s}{k+j+i+s}$$
$$\times (x-x_4)^{n+m-i-s} + K_4 \sum_{i=0}^{n} \binom{n}{i} \frac{(x_2-x_1)^i}{m+i} \sum_{s=0}^{n-i} \binom{n-i}{s} \frac{(x_3-x_2)^s}{m+k+i+s}$$
$$\times \sum_{t=0}^{n-s-i} \binom{n-s-i}{t} \frac{(x_4-x_3)^t}{m+k+j+s+i+t} (x-x_4)^{n-s-i-t}.$$

From this formula we can derive the solution in some special cases, that is when all or some of the roots of the equation $a(x)=0$ are equal.

References

[1] A b b é L a i n é. Sur l'integration de quelques équations différentielles du second ordre, *L'Ensiegnement mathématique* **23**, 1924, 163-173, Paris-Géneve.

[2] V. J. G o n s a l v e s. Sur la formule de Rodrigues, *Portugaliae Math.* **4**, 1934, 52-64.

[3] B. M. P i p e r e v s k i. Existence and construction of polynomial solution of a class of linear differential equations of third order, In: *Zbornik Trudove ETF, Skopje*, 1996 (In Macedonian).

[4] B. M. P i p e r e v s k i. Polynomial solutions of a class of linear differential equations, *Ph.D. Thesis*, Skopje, 1982 (In Macedonian).

[5] I. A. Š h a p k a r e v. A polynomial as a general solution of a class of linear differential equations, *Ph.D. Thesis*, Skopje, 2000 (In Macedonian).

[6] I. A. Š a p k a r e v. Sur une équation différentielle linéaire d'ordre n dont la solution générale est un polynom de n-éme degré, *Matematički Vesnik (Beograd)* **16**, No 1, 1964, 49-50.

[1,2] *Faculty of Electrical Engineering*
P.O. Box 574, 1000 Skopje, MACEDONIA

Received: 30.09.2003

[1] *e-mail: hadzieva@etf.ukim.edu.mk*