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One Subclass of Generalized Analytic Functions of Third Class

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In this paper generalized analytic functions of third class with characteristics $\mu = \lambda z^m$ ($\lambda \in \mathbf{C}$, $m \in \mathbf{N}_0$), defined by areolar differential equation, are considered via the so-called cylindrical functions of zero order.

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1. Introduction

G. V. Kolosov [1] in 1909, solving a problem of the elasticity theory, introduced the operator derivatives $\frac{\hat{d}}{dz}$ and $\frac{\hat{d}}{d\overline{z}}$ in $z = x + iy$ and $\overline{z} = x - iy$ with

$$\frac{\hat{d}w}{dz} = \frac{1}{2}\Big[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\Big] \tag{1}$$

and

$$\frac{\hat{d}w}{d\overline{z}} = \frac{1}{2}\Big[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)\Big], \tag{2}$$

where $w = w(z) = u(x,y) + iv(x,y)$. The operative derivative in $\overline{z} = x - iy$ (2) can be found in the scientific literature as a **areolar derivative**. The operation rules for these derivatives are given in the monograph of G. N. Polozhyi [2].

The double value of the areolar derivative

$$B = 2\frac{\hat{d}}{d\overline{z}}\,. \tag{3}$$

was used by A. Bilimović [3,4] as a measure of the deviation of analyticity of the nonanalytic functions.

S. Fempl [5],[6],[7] defined nonanalytic functions with first, second and n-th deviation of analytic function as solutions of the equation

$$B_n(w) = f(z) \tag{4}$$

for $n = 1$, $n = 2$ and $n \in \mathbf{N}$.

In the monograph [2], G. N. Polozhyi introduced some classes of generalized analytic functions. One of them is the class of generalized analytic functions of third class, defined as solutions of the areolar equation

$$\frac{\hat{d}w}{d\overline{z}} = \mu\overline{w}\,, \quad \mu = r + is. \tag{5}$$

As noted in Polozhyi [2], in a case of areolar equations of the 1-st order

$$\frac{\hat{d}w}{d\overline{z}} = f(z, w) \tag{6}$$

with analytic right-hand side $f = f(z,w)$ in $w = w(z)$ e.t. $\frac{\hat{d}f}{d\overline{w}} = 0$ (which is equivalent with the fact that in equation (6) the unknown function w is not under sign of complex conjugation) there exist algorithms for its quadrature solutions which are analogous with the algorithms for solutions of ordinary differential equations of the first order $y' = f(x, y)$, $y = f(x)$ and $x \in D \subset R$. In the role of integration constant, there appears an arbitrary analytic function of $z = x + iy$.

In the case when the right-hand side $f = f(z,w)$ in equation (6) is not analytic function of w, which means that in (6) the unknown function explicitly appears under the sign of complex conjugation, the analogous with the ordinary differential equations of the first order disappear.

From the above considerations, it follows that it is of interest to study the areolar equation

$$\frac{\hat{d}w}{d\overline{z}} = \lambda z^m \overline{w} \quad (\lambda \in \mathbf{C},\ m \in \mathbf{N}) \tag{7}$$

which is a special case of the Vekua equation

$$\frac{\hat{d}w}{d\overline{z}} = Aw + B\overline{w} + F$$

($A = 0$, $B = \lambda z^m$, $F = 0$) and referred to by Polozhyi [2] as analytic functions of characteristic $\mu = \lambda z^m$.

2. Main results

The papers [8], [9] and [10] contain for $m = 0$, $m = 1$ and $m = 2$ solutions of the areolar equation (7) in a form of areolar series

$$w = \sum_{p,q=0}^{\infty} C_{pq} z^p \overline{z}^q. \tag{8}$$

In the case $m = 0$, in [8] it is found that equation (7) has a solution

$$w = \phi(z) + \sum_{n=0}^{\infty} \frac{|\lambda|^{2(n+1)}}{(n+1)!} \overline{z}^{n+1} \int\int \cdots \int \phi(z)(dz)^{n+1}$$

$$+ \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}\lambda}{n!} z^n \int\int \cdots \int \overline{\phi(z)}(d\overline{z})^{n+1}. \tag{9}$$

In the case $m = 1$ (see [9]), equation (7) has a solution

$$w = \phi(z) + \lambda z \int \overline{\phi}(z)\, d\overline{z}$$

$$+ z \sum_{n=1}^{\infty} \frac{|\lambda|^{2n}}{2^n\, n!} \Big[\overline{z}^{2n} \underbrace{\int z\, dz \int z\, dz \int \cdots \int z\, dz \int}_{n-\text{integrals}} \phi(z) dz$$

$$+ \lambda z^{2n} \underbrace{\int \overline{z}\, d\overline{z} \int \overline{z}\, d\overline{z} \cdots \int \overline{z}\, d\overline{z} \int}_{n+1-\text{integrals}} \overline{\phi}(z)\, d\overline{z} \Big], \tag{10}$$

and in a case $m = 2$ the solution is:

$$w = \phi(z) + \lambda z^2 \int \overline{\phi}(z)\, d\overline{z}$$

$$+ z^2 \sum_{n=1}^{\infty} \frac{|\lambda|^{2n}}{3^n n!} \Big[\overline{z}^{3n} \underbrace{\int z^2\, dz \int z^2\, dz \int \cdots \int z^2\, dz \int}_{n-\text{integrals}} \phi(z) dz$$

$$+ \lambda z^{3n} \underbrace{\int \overline{z}^2\, d\overline{z} \int \overline{z}^2\, d\overline{z} \cdots \int \overline{z}^2\, d\overline{z} \int}_{n+1-\text{integrals}} \overline{\phi}(z)\, d\overline{z} \Big]. \tag{11}$$

In these cases in each of the formulas (9), (10) and (11),

$$\phi(z) = \sum_{p=0}^{\infty} C_{p_0} z^p$$

is an arbitrary analytic function playing the role of an integration constant.

Now, if we analyze formulas (9), (10) and (11) for the solutions of the areolar equation (7) in the cases $m = 0$, $m = 1$ and $m = 2$, then inductively for an arbitrary $m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ we can conclude that the equation (7) has a solution

$$w = \phi(z) + \lambda z^m \int \overline{\phi}(z)\, d\overline{z}$$

$$+z^m \sum_{n=1}^{\infty} \frac{|\lambda|^{2n}}{(m+1)^n\, n!} \Big[\overline{z}^{n(m+1)} \underbrace{\int z^m\, dz \int z^m\, dz \cdots \int z^m\, dz \int}_{n-\text{integrals}} \phi(z) dz$$

$$+\lambda z^{n(m-1} \underbrace{\int \overline{z}^m\, d\overline{z} \int \overline{z}^m\, d\overline{z} \cdots \int \overline{z}^m\, d\overline{z} \int}_{n+1-\text{integrals}} \overline{\phi}(z)\, d\overline{z}\Big]. \tag{12}$$

We state now the following problem:

Is it possible the general solution (12) of equation (7) to be written in the same form?

For this aim, we use the formula

$$\underbrace{\int z^m dz \int z^m dz \cdots \int z^m dz \int}_{n-\text{integrals}} \phi(z) dz = \begin{cases} \displaystyle\int \frac{(z-\zeta)^{(m+1)(n-1)}}{(m+1)^{n-1}(n-1)!} \phi(\zeta) d\zeta, & n \geq 2, \\ \displaystyle\int \phi(\zeta) d\zeta, & n = 1, \end{cases} \tag{13}$$

which is a kind of generalization of the known Cauchy theorem of n-times integration.

By formula (13), the solution (12) of equation (7) can be written in the form:

$$w = \phi(z) + \lambda z^m \int \overline{\phi}(z)\, d\overline{z}$$

$$+z^m \sum_{n=1}^{\infty} \frac{|\lambda|^{2n}}{(m+1)^n\, n!} \Big[\overline{z}^{n(m+1)} \int \frac{(z-\zeta)^{(m+1)(n-1)}}{(m+1)^{n-1}(n-1)!} \phi(\zeta) d\zeta \tag{14}$$

$$+\lambda z^{n(m+1)} \int \frac{(\overline{z}-\overline{\zeta})^{(m+1)n}}{(m+1)^n\, n!}\overline{\phi}(\zeta)d\overline{\zeta}\Big],$$

or in the form

$$w = \phi(z) + \lambda z^m \int \overline{\phi}(z)\, d\overline{z}$$

$$+z^m\Big\{\int\Big[\sum_{n=1}^{\infty}\frac{|\lambda|^{2n}(z-\zeta)^{(m+1)(n-1)}\overline{z}^{(m+1)n}}{(m+1)^n\, n!(m+1)^{n-1}(n-1)!}\Big]\phi(\zeta)d\zeta \tag{15}$$

$$+\lambda\int\Big[\sum_{n=1}^{\infty}\frac{|\lambda|^{2n}(\overline{z}-\overline{\zeta})^{(m+1)n}z^{(m+1)n}}{(m+1)^n\, n!(m+1)^n\, n!}\Big]\overline{\phi}(\zeta)d\overline{\zeta}\Big\},$$

where we exchange the operations $\sum$ and $\int$.

We note that the series in the middle part of (15) are uniformly convergent in each closed subdomain of the domain of analyticity of the function $\phi = \phi(t)$. So, it is possible from the form (14) to pass to the form (15).

Theorem 1. *The generalized analytic functions of third class with characteristics $\mu = r + is = \lambda z^m$, $m \in \mathbf{N}_0$ can be written in the forms (14) or (15), where $\phi = \phi(z)$ is an arbitrary holomorphic function.*

If we put

$$S = \sum_{n=1}^{\infty}\frac{|\lambda|^{2n}(\overline{z}-\overline{\zeta})z]^{(m+1)n}}{[(m+1)^n\, n!]^2}$$

$$= \sum_{n=1}^{\infty}\frac{[|\lambda|^{2/(m+1)}(\overline{z}-\overline{\zeta})z]^{(m+1)n}}{[(m+1)^n\, n!]^2} = \sum_{n=1}^{\infty}\frac{u^{(m+1)n}}{[(m+1)^n\, n!]^2} = S(u), \tag{16}$$

$$u = |\lambda|^{2/(m+1)}\,(\overline{z}-\overline{\zeta})z, \tag{17}$$

then we have

$$\frac{dS}{du} = \sum_{n=1}^{\infty}\frac{n(m+1)u^{n(m+1)-1}}{[(m+1)^n\, n!]^2},$$

$$u\frac{dS}{du} = \sum_{n=1}^{\infty}\frac{n(m+1)u^{n(m+1)}}{[(m+1)^n\, n!]^2}$$

and

$$\frac{d}{du}\left(u\frac{dS}{du}\right) = \sum_{n=1}^{\infty}\frac{n^2(m+1)^2u^{n(m+1)-1}}{[(m+1)^n\, n!]^2}$$

$$= u^m \sum_{n=1}^{\infty} \frac{u^{(n-1)(m+1)}}{[(m+1)^{n-1}(n-1)!]^2} = u^m(S(u)+1).$$

This means that the function $S = S(u)$, defined by (16), satisfies the differential equation

$$u\frac{d^2S}{du^2} + \frac{dS}{du} = u^m(S(u)+1)\,. \tag{18}$$

With the substitution

$$S(u) + 1 = T(u), \tag{19}$$

equation (18) can be transformed into

$$u\frac{d^2T}{du^2} + \frac{dT}{du} - u^m T = 0. \tag{20}$$

Note that in Kamke [11], page 401, the differential equation

$$x^2y'' + axy' + (bx^m + c)y = 0 \tag{21}$$

has a solution

$$y = \mathcal{Z}_0\left(\frac{2}{m+1} i x^{(m+1)/2}\right),$$

where $\mathcal{Z}_0$ is the so called cylindrical function of zero order.

Since the equation (20) is a special case of (21) ($a = 1$, $b = -1$, $c = 0$ and in the place of m we have $m+1$), we have

$$T(u) = \mathcal{Z}_0\left(\frac{2}{m+1} i u^{(m+1)/2}\right) \tag{22}$$

and in agreement with substitution (19),

$$S(u) = -1 + \mathcal{Z}_0\left(\frac{2}{m+1} i u^{(m+1)/2}\right),$$

and by (17):

$$S = -1 + \mathcal{Z}_0\left(\frac{2}{m+1} i|\lambda|((\overline{z} - \overline{\zeta})z)^{\frac{m+1}{2}}\right). \tag{23}$$

Then, we make summation of the second series in formula (15).

For the first series in formula (15) we have:

$$R=\sum_{n=1}^{\infty}\frac{|\lambda|^{2n}(z-\zeta)^{(m+1)(n-1)}\overline{z}^{n(m+1)}}{(m+1)^n\, n!\,(m+1)^{n-1}(n-1)!} \tag{24}$$

and

$$\begin{aligned}\frac{\hat{d}R}{d\overline{z}}&=\sum_{n=1}^{\infty}\frac{|\lambda|^{2n}(z-\zeta)^{(m+1)(n-1)}\, n(m+1)\overline{z}^{n(m+1)-1}}{(m+1)^n\, n!(m+1)^{n-1}(n-1)!}\\&=\sum_{n=1}^{\infty}\frac{|\lambda|^{2n}(z-\zeta)^{(m+1)(n-1)}\overline{z}^{n(m+1)-1}}{[(m+1)^{n-1}(n-1)!]^2}\\&=\sum_{n=1}^{\infty}\frac{|\lambda|^{2n}(z-\zeta)^{(m+1)(n-1)}\overline{z}^{m}\overline{z}^{(m+1)(n-1)}}{[(m+1)^{n-1}(n-1)!]^2}\\&=|\lambda|^2\overline{z}^m\sum_{n=1}^{\infty}\frac{[|\lambda|^{2/(m+1)}(z-\zeta)\overline{z}]^{(m+1)(n-1)}}{[(m+1)^{n-1}(m-1)!]^2}\,,\end{aligned}$$

and using (17), (16) and (23), we obtain:

$$\frac{\hat{d}R}{d\overline{z}}=|\lambda|^2\overline{z}^m\sum_{n=1}^{\infty}\frac{\overline{u}^{(m+1)(n-1)}}{[(m+1)^{n-1}(n-1)!]^2}$$

$$=|\lambda|^2\overline{z}^m[S(\overline{u})+1]=|\lambda|^2\overline{z}^m\left[-1+\mathcal{Z}_0\left(\frac{2}{m+1}i|\lambda|\,((z-\zeta)\overline{z})^{\frac{m+1}{2}}\right)+1\right].$$

From the above, we get

$$R=|\lambda|^2\widehat{\int}\overline{z}^m\mathcal{Z}_0\left(\frac{2}{m+1}i|\lambda|\,((z-\zeta)\overline{z})^{\frac{m+1}{2}}\right)d\overline{z}\,. \tag{25}$$

If we put (23) and (24) in (15), we get

$$w=\phi(z)+\lambda z^m\int\mathcal{Z}_0\left(\frac{2}{m+1}i|\lambda|\,((\overline{z}-\overline{\zeta})z)^{\frac{m+1}{2}}\right)\overline{\phi}|(\zeta)d\overline{\zeta}$$

$$+\int\left[|\lambda|^2\widehat{\int}|z|^{2m}\mathcal{Z}_0\left(\frac{2}{m+1}i|\lambda|\,((z-\zeta)\overline{z})^{\frac{m+1}{2}}\right)d\overline{z}\right]\phi(\zeta)d\zeta\,. \tag{26}$$

Theorem 2. *The generalized functions of third class with characteristics $\mu=\lambda z^m$ $(m\in\mathbf{N}_0)$ can be written in terms of the so called cylindrical functions of zero order by formula (26).*

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