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On Some Properties of Complex Harmonic Mappings with a Two-Parameter Coefficient Condition

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In 1984 J. Clunie and T. Sheil-Small initiated studies of complex functions harmonic in the unit disc. Several mathematicians examined classes of complex mappings harmonic in the unit disc with some coefficient conditions, e.g. [1], [2], [4], [6], [7], [8]. We present some results which generalize problems considered, among others, in the papers [4] and [6]. The main results concern starlikeness and convexity of harmonic functions satisfying a two-parameter coefficient condition.

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Let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$, $\mathbf{P} = \{(\alpha, p) \in \mathbf{R}^2 : 0 \leq \alpha \leq 1, p > 0\}$,

$$U_n(\alpha, p) = \alpha n^p + (1-\alpha) n^{p+1}, \quad n = 2, 3, \ldots, \ (\alpha, p) \in \mathbf{P}.$$

For a fixed pair $(\alpha, p) \in \mathbf{P}$ we denote by $HS(\alpha, p)$ the class of functions f of the form

$$(1)\quad f = h + \overline{g}, \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \ |b_1| < 1,$$

and such that

$$(2)\qquad |b_1| + \sum_{n=2}^{\infty} U_n(\alpha, p) \left(|a_n| + |b_n|\right) \leq 1.$$

Moreover,

$$HS^0(\alpha, p) = \{f \in HS(\alpha, p) : b_1 = 0\}.$$

The classes $HS(1,1)$ $HS^0(1,1)$, $HS(1,2)$ $HS^0(1,2)$ were investigated in paper [2], the classes $HS(1,p)$, $HS^0(1,p)$ $(p > 0)$ were considered in [4] and the classes $HS(\alpha, 1)$, $HS^0(\alpha, 1)$ for $\alpha \in \langle 0, 1\rangle$ - in [6].

It is known that each function of the class $HS^0(1,1)$ is starlike and every function of the class $HS^0(1,2)$ is convex ([2]).

With respect to the following inequalities

$$U_n(1,p) = n^p \leq U_n(\alpha, p) \leq n^{p+1} = U_n(0,p), \quad n = 2, 3, \ldots, \ (\alpha, p) \in \mathbf{P},$$

and by (2) we have the obvious inclusions

$$HS(0,p) \subset HS(\alpha, p) \subset HS(1,p), \quad (\alpha, p) \in \mathbf{P},$$

$$HS^0(0,p) \subset HS^0(\alpha, p) \subset HS^0(1,p), \quad (\alpha, p) \in \mathbf{P}. \tag{3}$$

Directly from the definition of the class $HS(\alpha, p)$ $(HS^0(\alpha, p))$ we get the following.

Theorem 1. *Let $(\alpha, p) \in \mathbf{P}$. If $f \in HS(\alpha, p)$ $(HS^0(\alpha, p))$, then functions*

$$z \mapsto r^{-1} f(rz), \qquad z \mapsto e^{-it} f(e^{it} z), \qquad z \in \Delta, \ r \in (0,1), \ t \in \mathbf{R},$$

also belong to $HS(\alpha, p)$ (resp. $HS^0(\alpha, p)$).

Theorem 2. *If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $p > 0$, then*

$$HS(\alpha_1, p) \subset HS(\alpha_2, p), \qquad HS^0(\alpha_1, p) \subset HS^0(\alpha_2, p).$$

If $\alpha \in \langle 0, 1\rangle$ and $0 < p_1 \leq p_2$, then

$$HS(\alpha, p_1) \supset HS(\alpha, p_2), \qquad HS^0(\alpha, p_1) \supset HS^0(\alpha, p_2).$$

Remark 1. Let $(\alpha, p) \in \mathbf{P}$. If $f \in HS(\alpha, p)$, f of the form (1), then the function f_0 of the form

$$f_0(z) = \frac{f(z) - \overline{b_1 f(z)}}{1 - |b_1|^2}\ , \qquad z \in \Delta \quad (|b_1| < 1),$$

belongs to the class $HS^0(\alpha,p)$. However, if $f_0 \in HS^0(\alpha,p)$ and $|b_1| < 1$, the function of the form

$$f(z) = f_0(z) + \overline{b_1 f_0(z)}, \qquad z \in \Delta,$$

does not have to belong to $HS(\alpha,p)$.

Example. Let $(\alpha,p) \in \mathbf{P}$ and

$$f_1(\alpha,p;z) = z + \frac{1}{U_2(\alpha,p)} z^2, \qquad z \in \Delta. \tag{4}$$

We can see that $f_1(\alpha,p;\cdot) \in HS^0(\alpha,p)$, of course. Moreover, $f_1(1,p;\cdot) \notin HS^0(\alpha,p)$ for $\alpha \in \langle 0,1)$ and $f_1(\alpha,p;\cdot) \notin HS^0(0,p)$ for $\alpha \in (0,1\rangle$. It means that the inclusions (3) are strict.

We can check that for $0 < |b_1| < 1$ the function $z \mapsto f_1(\alpha,p;z) + \overline{b_1 f_1(\alpha,p;z)}$, $z \in \Delta$, does not belong to the class $HS(\alpha,p)$.

Similarly as in the papers [2], [4] we obtain:

Theorem 3. *Let $(\alpha,p) \in \mathbf{P}$. If $p \geq 1$, then every function $f \in HS^0(\alpha,p)$ is univalent and maps the unit disc Δ onto a domain starlike with respect to the origin (f is a starlike function).*

If $p \geq 2$, then each function $f \in HS^0(\alpha,p)$ is univalent and maps Δ onto a convex domain (f is convex).

P r o o f. If $p \geq 1$, then $U_n(\alpha,p) \geq n$ for $n = 2,3,\ldots$, $\alpha \in \langle 0,1\rangle$, so by the condition (2) we obtain

$$\sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right) \leq 1.$$

Therefore (see [2]), f is univalent and starlike with respect to the origin. If $p \geq 2$, then by (2) we get

$$\sum_{n=2}^{\infty} n^2\left(|a_n| + |b_n|\right) \leq 1.$$

Hence ([2]), f is convex. ■

Next, let $\alpha \in \langle 0,1\rangle$ and set $p_1(\alpha) = 1 - log_2(2-\alpha)$, $p_2(\alpha) = 2 - log_2(2-\alpha)$, $log_2 1 = 0$. We denote

$$D_1 = \{(\alpha,p) \in \mathbf{P} : p \geq p_1(\alpha)\},$$

$$D_2 = \{(\alpha, p) \in \mathbf{P} : p \geq p_2(\alpha)\}.$$

The next theorems present results concerning starlikeness and convexity of functions of the class $HS^0(\alpha, p)$ for arbitrary $(\alpha, p) \in D_1$ and $(\alpha, p) \in D_2$, respectively.

Theorem 4. *If $(\alpha, p) \in D_1$, then the functions of the class $HS^0(\alpha, p)$ are starlike. If $(\alpha, p) \in D_2$, then the functions of the class $HS^0(\alpha, p)$ are convex.*

Proof. We can check that the following inequalities

$$U_n(\alpha, p) \geq n, \qquad (\alpha, p) \in D_1, \quad n = 2, 3, \ldots,$$

$$U_n(\alpha, p) \geq n^2, \qquad (\alpha, p) \in D_2, \quad n = 2, 3, \ldots$$

hold. If $f \in HS^0(\alpha, p)$ for $(\alpha, p) \in D_1$, then in view of the first inequality, the condition (2) and of the mentioned result from [2] it follows that f is a starlike function. Analogously, from the second inequality, $f \in HS^0(\alpha, p)$ for $(\alpha, p) \in D_2$ is convex ([2]). ■

Naturally, in view of Theorem 3 it suffices to examine the cases $p < 1$ and $p < 2$, respectively.

Next, let $\Delta_r = \{z \in \mathbf{C} : |z| < r\}$, $r > 0$, with $\Delta_1 = \Delta$. Moreover, we prove the following

Theorem 5. *Let $(\alpha, p) \in \mathbf{P} \setminus D_1$. If $r \in (0, r_0^*(\alpha, p)\rangle$, where $r_0^*(\alpha, p) = 2^{p-1}(2 - \alpha)$, then each function $f \in HS^0(\alpha, p)$ maps the disc Δ_r onto a domain starlike with respect to the origin. The constant $r_0^*(\alpha, p)$ is the best possible.*

Proof. For $(\alpha, p) \in \mathbf{P} \setminus D_1$ we have $r_0^*(\alpha, p) < 1$, of course. Let $f \in HS^0(\alpha, p)$, $(\alpha, p) \in \mathbf{P} \setminus D_1$ and let $r \in (0, r_0^*(\alpha, p)\rangle$. By Theorem 1 the function f_r of the form $f_r(z) = r^{-1} f(rz)$ belongs to the class $HS^0(\alpha, p)$ and we have

$$\sum_{n=2}^{\infty} n \left(|a_n r^{n-1}| + |b_n r^{n-1}| \right) = \sum_{n=2}^{\infty} n r^{n-1} \left(|a_n| + |b_n| \right).$$

In view of properties of elementary functions we obtain

$$n r^{n-1} \leq n (r_0^*(\alpha, p))^{n-1} \leq U_n(\alpha, p), \quad n = 2, 3, \ldots.$$

Hence $f_r \in HS^0(1, 1)$ ([2]) for any $r \in (0, r_0^*(\alpha, p)\rangle$ maps Δ onto a domain starlike with respect to the origin. From this fact we get the assertion.

The constant $r_0^*(\alpha, p)$ cannot be improved because of e.g. the function $f_1(\alpha, p; \cdot)$ of the form (4), which belongs to the class $HS^0(\alpha, p)$ (see Example). ■

Theorem 6. *Let* $(\alpha,p)\in \mathbf{P}\setminus D_2$ *and set* $r_0(\alpha,p)=2^{p-2}(2-\alpha)$. *If* $r\in(0,r_0(\alpha,p)\rangle$, *then each function* $f\in HS^0(\alpha,p)$ *maps* Δ_r *onto a convex domain. The constant* $r_0(\alpha,p)$ *is the best possible.*

P r o o f. For every $(\alpha,p)\in \mathbf{P}\setminus D_2$ we have $r_0(\alpha,p)<1$. Further we proceed analogously as in the proof of Theorem 5, examining that for any $r\in (0,r_0(\alpha,p)\rangle$

$$n^2 r^{n-1}\le U_n(\alpha,p),\quad n=2,3,\ldots.$$

Similarly, an extremal function is also the function $f_1(\alpha,p;\cdot)$ of the form (4). ■

Remark 2. One can easily check that the classes $HS(\alpha,p)$, $HS^0(\alpha,p)$, $(\alpha,p)\in\mathbf{P}$, are convex.

By the condition (2) we have also the following

Theorem 7. *Let* $(\alpha,p)\in\mathbf{P}$. *If* $f\in HS(\alpha,p)$,$z\in\Delta$, $z\neq 0$, *then*

$$|f(z)|\le(1+|b_1|)\,|z|+\frac{1-|b_1|}{2^p(2-\alpha)}|z|^2,$$

$$|f(z)|\ge(1-|b_1|)\,|z|-\frac{1-|b_1|}{2^p(2-\alpha)}|z|^2.$$

The estimates are sharp.

P r o o f. Let $f\in HS(\alpha,p)$, $(\alpha,p)\in\mathbf{P}$, f of the form (1), and fix $z\in\Delta\setminus\{0\}$. Then the condition (2) holds and after some simple transformations we obtain

$$\sum_{n=2}^{\infty}(|a_n|+|b_n|)\le\frac{1-|b_1|}{U_2(\alpha,p)}-\sum_{n=3}^{\infty}\left(\frac{U_n(\alpha,p)}{U_2(\alpha,p)}-1\right)(|a_n|+|b_n|)\,.$$

Since $U_n(\alpha,p)\ge U_2(\alpha,p)$, $n=3,4,\ldots$, $(\alpha,p)\in\mathbf{P}$, we have

$$\sum_{n=2}^{\infty}(|a_n|+|b_n|)\le\frac{1-|b_1|}{U_2(\alpha,p)}\,.$$

Hence

$$|f(z)|\le\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n+(1+|b_1|)\,|z|\le(1+|b_1|)\,|z|+\frac{1-|b_1|}{U_2(\alpha,p)}|z|^2,$$

that is the upper estimate.

The lower estimate follows from the inequality

$$|f(z)| \geq |z| - |b_1||z| - \sum_{n=2}^{\infty} (|a_n| + |b_n|) |z|^n$$

and from the previous inequality for the coefficients.

Extremal functions are e.g. the functions of the form $z \mapsto z + b_1\overline{z} + \frac{1-|b_1|}{2^p(2-\alpha)}z^2$, $z \mapsto z + b_1\overline{z} + \frac{1-|b_1|}{2^p(2-\alpha)}\overline{z^2}$, $|b_1| < 1$, $z \in \Delta$. ∎

From the first part of Theorem 7 we obtain the following

Corollary. *For $(\alpha, p) \in \mathbf{P}$ the class $HS(\alpha, p)$ is a normal family.*

Remark 3. It can be shown that the class $HS(\alpha, p)$, $(\alpha, p) \in \mathbf{P}$, is not compact (see [2], [6]).

However, we can prove

Theorem 8. *Let $(\alpha, p) \in \mathbf{P}$. The class $HS^0(\alpha, p)$ is compact and the extreme points of the class $HS^0(\alpha, p)$ are only the functions $z \mapsto z + a_m z^m$, $\mapsto z + \overline{b_k z^k}$, $z \in \Delta$, where $|a_m| = \frac{1}{U_m(\alpha,p)}$, $|b_k| = \frac{1}{U_k(\alpha,p)}$, $m, k = 2, 3, \ldots$.*

P r o o f. Let $(\alpha, p) \in \mathbf{P}$ and let $\{f_k\}$, $f_k \in HS^0(\alpha, p)$, where $f_k = h_k + \overline{g_k}$, $h_k(0) = h_k'(0) - 1 = 0$, $g_k(0) = g_k'(0) - 1 = 0$, $k = 1, 2, \ldots$, be a sequence uniformly convergent on compact subsets of Δ to a function f. Proceeding similarly as in [6], we prove that $f = h + \overline{g}$, where h, g are holomorphic in Δ, $h(0) = h'(0) - 1 = 0$, $g(0) = g'(0) - 1 = 0$. Therefore f is harmonic in Δ and is of the form (1) with $b_1 = 0$. We can easily check that f satisfies the condition (2), because the functions f_k, $k = 1, 2, \ldots$, satisfy it.

The forms of the extremal points of the class $HS^0(\alpha, p)$, $(\alpha, p) \in \mathbf{P}$, we determine in an analogous way as in [6]. ∎

The presented theorems generalize the respective results from the papers [4], [6], [2]. The main idea of the research included in this paper and other publications of this kind comes from the classical conditions, that imply starlikeness and convexity of functions holomorphic in the disc Δ (see [5]).

References

[1] G. A d a m c z y k, A. Ł a z i ń s k a. On some generalization of coefficient conditions for complex harmonic mappings, To appear in: *Demonstratio Mathematica*.

[2] Y. A v c i, E. Z ł o t k i e w i c z. On harmonic univalent mappings, *Ann. Univ. Mariae Curie-Skłodowska, Sec. A* **XLIV (1)**, 1990, 1-7.

[3] J. C l u n i e, T. S h e i l - S m a l l. Harmonic univalent mappings, *Ann. Acad. Sci. Fenn., Ser. A. I. Math.* **9**, 1984, 3-25.

[4] A. G a n c z a r. On harmonic univalent functions with small coefficients, *Demonstratio Math.* **XXXIV (3)**, 2001, 549-558.

[5] A. G o o d m a n. Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.* **8**, 1957, 588-601.

[6] A. Ł a z i ń s k a. On complex mappings harmonic in the unit disc with some coefficient conditions, *Folia Sci. Univ. Technicae Resoviensis, Mat. Z.* **26**, **199**, 2002, 107-116.

[7] H. S i l v e r m a n. Harmonic univalent mappings with negative coefficients, *J. Math. Anal. Appl.* **220**, 1998, 283-289.

[8] J. M. Y a h a n g i r i. Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, *Ann. Univ. Mariae Curie-Sklodowska, Sec. A* **LII (2)**, 1998, 57-66.

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