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L^{2m}-Asymptotic Behavior of Solutions of Perturbed Stochastic Integrodifferential Equations [1]

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This paper is devoted to a very general class of stochastic integrodifferential equations whose coefficients are perturbed and depend on a small parameter. We expose conditions under which the solutions of the perturbed and the corresponding unperturbed equations are close in the $(2m)$-th moment sense, on intervals whose length tends to infinity as a small parameter tends to zero. We apply two different methods to determine these intervals and we compare their lengths and degree of the closeness for the solutions on these intervals.

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1. Introduction

Stochastic differential equations play a mayor role in the mathematical characterization of many real phenomena in life and natural sciences, as well as in engineering, depending on the effect of "white noise" excitations. Especially, if a random excitation is a Gaussian white noise, which is only a fiction and not a real process, mathematically described as a formal derivative of a Brownian motion process, all such phenomena are mathematically modelled by various classes of stochastic differential equations of the Ito type, depending on parameters and perturbations. In fact, in the present paper we consider the problem of perturbations for one of very general stochastic integrodifferential equation,

$$(1)\qquad dx_t = \left[a_1(t,x_t) + \int_0^t a_2(t,s,x_s)\,ds + \int_0^t a_3(t,s,x_s)\,dw_s\right] dt$$

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$$+\left[b_1(t,x_t)+\int_0^t b_2(t,s,x_s)\,ds+\int_0^t b_3(t,s,x_s)\,dw_s\right]dw_t,\quad t\in[0,T],$$
$$x(0)=x_0\quad \text{a.s.}$$

studied earlier in details in papers [2, 3] by Berger and Mizel, and generalized later in many papers, in [7, 12, 13], for example. Here $w=(w_t,t\in R)$ is an R^k-valued normalized Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},P)$, with a natural filtration $\{\mathcal{F}_t,t\geq 0\}$ ($\mathcal{F}_t=\sigma\{x_s,0\leq s\leq t\}$) of nondecreasing sub σ-algebras of $\mathcal{F}$, x_0 is an R^n-random variable independent of w, the functions

$$\begin{array}{ll} a_1:[0,T]\times R^n\to R^n, & b_1:[0,T]\times R^n\to R^n\times R^k,\\ a_2:J\times R^n\to R^n, & b_2:J\times R^n\to R^n\times R^k,\\ a_3:J\times R^n\to R^n\times R^k, & b_3:J\times R^n\to R^n\times R^k\times R^k,\end{array}$$

where $J=\{(t,s)\,:\,0\leq s\leq t\leq T\}$, are Borel measurable on their domains, and $x=(x_t,t\in[0,T])$ is an R^n-valued unknown stochastic process. The process x is a strong solution of Eq. (1) if it is adapted to $\{\mathcal{F}_t,t\geq 0\}$, $\int_0^T|a_1(t,x_t)|\,dt<\infty$ a.s., $\int_0^T|b_1(t,x_t)|^2\,dt<\infty$ a.s., $\int_0^T\int_0^t|a_2(t,s,x_s)|\,dsdt<\infty$ a.s., $\int_0^T\int_0^t|f(t,s,x_s)|^2\,dsdt<\infty$ a.s. for $a_3(\cdot),b_2(\cdot),b_3(\cdot)$ instead of f (under these conditions all Lebesgue and Ito integrals in the integral form of Eq. (1) are well defined; $|\cdot|$ is an appropriate Euclidean matrix norm), and Eq. (1) holds a.s. for each $t\in[0,T]$.

On the basis of the classical theory of stochastic differential equations (see [4, 5, 9, 10], for example), the following existence and uniqueness assertion is proved in paper [2]: If the functions $a_i,b_i,\ i=1,2,3$, satisfy the global Lipschitz condition and the usual linear growth condition on the last argument, i.e. if there exists a constant $L>0$, so that, for all $(t,s)\in J$, $x,y\in R^n$,

$$(2)\qquad |a_2(t,s,x)-a_2(t,s,y)|<L\,|x-y|,\quad |a_2(t,s,x)|\leq L\,(1+|x|),$$

and similarly for the other functions, and if $E|x_0|^2<\infty$, then there exists a unique a.s. continuous strong solution $x=(x_t,t\in[0,T])$ of Eq.(1), satisfying $E\{\sup_{t\in[0,T]}|x_t|^2\}<\infty$. Moreover, following the procedures in [4, 10], it can be proved that if $E|x_0|^{2m}<\infty$ for a positive integer m, then $E\{\sup_{t\in[0,T]}|x_t|^{2m}\}<\infty$.

Bearing in mind that a class of effectively solvable Eq. (1) is small, from theoretical point of view and from various applications, it is important to compare, in some sense, its solution with a solution of an appropriate simpler equation, which will be the object of our investigation. In fact, in the present paper we improve results of paper [6], in which Eq. (1) depending on perturbations and on a small parameter is considered. Its solution is compared in the $(2m)$-th moment sense, with the solution of an equation of the same type, independent of perturbations. Note that perturbed stochastic differential and integrodifferential equations have been studied by many authors for ages. The basic types of perturbations considered here go back to papers [8, 11, 14], first of all to [14].

The paper is organized in the following way: In the next section we briefly give some notations and known results from paper [6], i.e. we impose intervals on which the solutions of the perturbed and unperturbed equations are close in the $(2m)$-th moment sense, which length tends to infinity as a small parameter tends to zero. In Section 3, which contains main results, we find intervals of the $(2m)$-th moment closeness of these solutions, whose lengths are much larger, and the degree of the closeness is much smaller than the ones from paper [6].

2. Preliminary results

In this section some results of paper [6] will be described in short, by expressing them as one assertion.

Together with Eq.(1) in equivalent integral form,

$$\begin{aligned} x_t = x_0 + \int_0^t \left[a_1(s, x_s) + \int_0^s a_2(s, r, x_r)\, dr + \int_0^s a_3(s, r, x_r)\, dw_r \right] ds \\ + \int_0^t \left[b_1(s, x_s) + \int_0^s b_2(s, r, x_r)\, dr + \int_0^s b_3(s, r, x_r)\, dw_r \right] dw_s, \end{aligned} \tag{3}$$

the following stochastic integrodifferential equation depending on a small parameter $\varepsilon \in (0, 1)$,

$$\begin{aligned} x_t^\varepsilon = x_0^\varepsilon + \int_0^t \left[\tilde{a}_1(s, x_s^\varepsilon, \varepsilon) + \int_0^s \tilde{a}_2(s, r, x_r^\varepsilon, \varepsilon)\, dr + \int_0^s \tilde{a}_3(s, r, x_r^\varepsilon, \varepsilon)\, dw_r \right] ds \\ + \int_0^t \left[\tilde{b}_1(s, x_s^\varepsilon, \varepsilon) + \int_0^s \tilde{b}_2(s, r, x_r^\varepsilon, \varepsilon)\, dr + \int_0^s \tilde{b}_3(s, r, x_r^\varepsilon, \varepsilon)\, dw_r \right] dw_s \end{aligned} \tag{4}$$

was earlier studied in paper [6]. Here $t \in [0, T]$, the initial condition x_0^ε is an R^n-valued random variable independent of w, the coefficients $\tilde{a}_i, \tilde{b}_i, i = 1, 2, 3$ are defined with the help of a_i, b_i and of the functions $\alpha_i, \beta_i, i = 1, 2, 3$, depending on ε, so that, for $(t, s) \in J$, $x \in R^n$,

$$\begin{aligned} &\tilde{a}_1(t, x, \varepsilon) = a_1(t, x) + \alpha_1(t, x, \varepsilon), \\ &\tilde{b}_1(t, x, \varepsilon) = b_1(t, x) + \beta_1(t, x, \varepsilon), \\ &\tilde{a}_i(t, s, x, \varepsilon) = a_i(t, s, x) + \alpha_i(t, s, x, \varepsilon), \quad i = 2, 3, \\ &\tilde{b}_i(t, s, x, \varepsilon) = b_i(t, s, x) + \beta_i(t, s, x, \varepsilon), \quad i = 2, 3. \end{aligned}$$

In accordance with [6, 7], and first of all with [14], the functions α_i and β_i are called *the perturbations* of Eq. (3), while Eq. (4) is logically called *the perturbed equation* with respect to *the unperturbed equation* (3). Likewise, in paper [6] the following assumptions were introduced:

For a positive integer m, let there exist a non-random values $\delta_0(\varepsilon)$ $\delta_i(\varepsilon)$, $\gamma_i(\varepsilon)$, $i = 1, 2, 3$, such that

$$(5)\quad E|x_0|^{2m} < \infty, \quad E|x_0^\varepsilon|^{2m} < \infty, \quad E|x_0 - x_0^\varepsilon|^{2m} \le \delta_0(\varepsilon),$$

$$(6)\quad \begin{aligned} &\sup_{(t,x)\in[0,T]\times R^n} |\alpha_1(t,x,\varepsilon)| \le \delta_1(\varepsilon), \qquad \sup_{(t,x)\in[0,T]\times R^n} |\beta_1(t,x,\varepsilon)| \le \gamma_1(\varepsilon),\\ &\sup_{(t,s,x)\in J\times R^n} |\alpha_i(t,s,x,\varepsilon)| \le \delta_i(\varepsilon), \qquad \sup_{(t,s,x)\in J\times R^n} |\beta_i(t,s,x,\varepsilon)| \le \gamma_i(\varepsilon),\ i = 2, 3.\end{aligned}$$

We also suppose without special emphasizing, that the all random and non-random integrals employed further are well defined, as well as that there exist unique solutions of equations (3) and (4), satisfying $E\{\sup_{t\in[0,T]} |x_t|^{2m}\} < \infty$ and $E\{\sup_{t\in[0,T]} |x_t^\varepsilon|^{2m}\} < \infty$. Furthermore, we shall emphasize only the conditions immediately used in our investigation.

If $\delta_0(\varepsilon), \delta_i(\varepsilon), \gamma_i(\varepsilon), i = 1, 2, 3$, tend to zero as ε tends to zero, it is reasonable to expect that the solution x^ε tends to x in some sense. The closeness in the $(2m)$-th moment sense of these solutions in paper [6] was proved. Here we shall express this assertion by the following theorem:

Theorem 1. *(i) Let x and x^ε be the solutions of the equations* (3) *and* (4) *respectively, the conditions* (2), (5) *and* (6) *be satisfied on a finite time-interval* $[0, T]$, *and* $\delta_0(\varepsilon)$, $\delta_i(\varepsilon)$, $\gamma_i(\varepsilon)$, $i = 1, 2, 3$, *tend to zero as ε tends to zero. Then*

$$\sup_{t\in[0,T]} E|x_t^\varepsilon - x_t|^{2m} \to 0 \quad as \quad \varepsilon \to 0.$$

(ii) Let the conditions from (i) be satisfied for $t \ge 0$. Then, for an arbitrary number $\overline{r} \in (0, 1)$ and ε sufficiently small, there exists a number $\overline{T}(\varepsilon) > 0$,

$$(7)\quad \overline{T}(\varepsilon) = M\left[\left(-\,\overline{r}\ln\overline{\phi}(\varepsilon)\right)^{1/4} - K\right]^{1/m},$$

where

$$(8)\quad \overline{\phi}(\varepsilon) = \max\left\{\delta_0(\varepsilon), \delta_i^{2m}(\varepsilon), \gamma_i^{2m}(\varepsilon), i = 1, 2, 3\right\},$$

and M and K are some generic positive constants, so that

$$\sup_{t\in[0,\overline{T}(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \to 0 \quad as \quad \varepsilon \to 0.$$

In order to estimate $E|x_t^\varepsilon - x_t|^{2m}$ for $t \in [0,T]$, let us denote that

$$z_t^\varepsilon = x_t^\varepsilon - x_t, \qquad \Delta_t^\varepsilon = E|z_t^\varepsilon|^{2m}, \quad t \in [0,T]. \tag{9}$$

If we subtract the equations (3) and (4) and apply the elementary inequality $\left(\sum_{i=1}^n a_i\right)^s \le n^{s-1}\sum_{i=1}^n a_i^s,\ a_i \ge 0, s \in N$, we obtain, for every $t \in [0,T]$,

$$\begin{aligned}
\Delta_t^\varepsilon \le 13^{2m-1}\Bigg[&E|z_0^\varepsilon|^{2m} + E\Big(\int_0^t [a_1(s,x_s^\varepsilon) - a_1(s,x_s)]\,ds\Big)^{2m}\\
&+ E\Big(\int_0^t \alpha_1(s,x_s^\varepsilon,\varepsilon)\,ds\Big)^{2m}\\
&+ E\Big(\int_0^t\int_0^s [a_2(s,r,x_r^\varepsilon) - a_2(s,r,x_r)]\,dr\,ds\Big)^{2m}\\
&+ E\Big(\int_0^t\int_{t_0}^s \alpha_2(s,r,x_r^\varepsilon,\varepsilon)\,dr\,ds\Big)^{2m}.\\
&+ E\Big(\int_0^t\int_0^s [a_3(s,r,x_r^\varepsilon) - a_3(s,r,x_r)]\,dw_r\,ds\Big)^{2m}\\
&+ E\Big(\int_0^t\int_0^s \alpha_3(s,r,x_r^\varepsilon,\varepsilon)\,dw_r\,ds\Big)^{2m}\\
&+ E\Big(\int_0^t [b_1(s,x_s^\varepsilon) - b_1(s,x_s)]\,dw_s\Big)^{2m}\\
&+ E\Big(\int_0^t \beta_1(s,x_s^\varepsilon,\varepsilon)\,dw_s\Big)^{2m}\\
&+ E\Big(\int_0^t\int_0^s [b_2(s,r,x_r^\varepsilon) - b_2(s,r,x_r)]\,dr\,dw_s\Big)^{2m}\\
&+ E\Big(\int_0^t\int_{t_0}^s \beta_2(s,r,x_r^\varepsilon,\varepsilon)\,dr\,dw_s\Big)^{2m}\\
&+ E\Big(\int_0^t\int_0^s [b_3(s,r,x_r^\varepsilon) - b_3(s,r,x_r)]\,dw_r\,dw_s\Big)^{2m}\\
&+ E\Big(\int_0^t\int_0^s \beta_3(s,r,x_r^\varepsilon,\varepsilon)\,dw_r\,dw_s\Big)^{2m}\Bigg].
\end{aligned} \tag{10}$$

By applying the usual stochastic integral isometry for the Lebesgue and Ito integrals in the previous relation, it was finally showed that

$$\Delta_t^\varepsilon \le \overline{\phi}(\varepsilon)\cdot P_4(t^m)\cdot e^{\zeta(t^m)}, \quad t \in [0,T],$$

where $\overline{\phi}(\varepsilon)$ is given by (8), $P_4(u) = \sum_{k=0}^4 c_k u^k$ and $\zeta(u) = u \cdot \sum_{k=0}^3 d_k u^k$ are polynomials with some generic coefficients c_k, d_k. Because the time interval is finite and $\overline{\phi}(\varepsilon) \to 0$ as $\varepsilon \to 0$, then $\sup_{t\in[0,T]} \Delta_t^\varepsilon \to 0$ as $\varepsilon \to 0$.

Since the assertion (i) is not generally valid for $t \geq 0$, we determine intervals $[0, \overline{T}(\varepsilon)]$ on which the solutions x^ε and x are close in the $(2m)$-th moment sense. In fact, $\overline{T}(\varepsilon)$ is determined from the relation $\zeta(\overline{T}^m(\varepsilon)) \leq -\overline{r} \ln \overline{\phi}(\varepsilon)$, where $\overline{r} \in (0,1)$ is an arbitrary number. Finally, we obtain $\overline{T}(\varepsilon)$ in the form (7) and, therefore,

$$\Delta_t^\varepsilon \leq \overline{\phi}(\varepsilon)^{1-\overline{r}} \cdot P_4(\overline{T}^m(\varepsilon)) \equiv \overline{R}(\varepsilon), \quad t \in [0, \overline{T}(\varepsilon)], \tag{11}$$

for $0 < \varepsilon \leq \overline{\varepsilon}_0 < 1$, where $\overline{\varepsilon}_0$ is such that $\overline{\phi}(\varepsilon) < 1$ for $\varepsilon \in (0, \overline{\varepsilon}_0)$. Thus, $\sup_{t\in[0,\overline{T}(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \leq \overline{R}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Therefore, the length of the intervals $[0, \overline{T}(\varepsilon)]$ tends to infinity as $\varepsilon \to 0$, and the rate of the closeness of the solutions x^ε and x tends to zero as $\varepsilon \to 0$.

3. Main results

In this section, under the conditions of Theorem 1 we shall estimate Δ_t^ε by applying Ito differential formula, which will give us a possibility to find intervals $[0, T(\varepsilon)]$, much larger than $[0, \overline{T}(\varepsilon)]$ from the previous section, so that the solutions x and x^ε of the equations (3) and (4) are close in the $(2m)$-th moment sense on these intervals. Note that Eq. (4) is a special case of a perturbed stochastic integrodifferential equation considered in paper [7], and, therefore, the proof of the next theorem will be partially similar to the ones from this paper.

Theorem 2. *Let the conditions of Theorem 1 be satisfied for $t \geq 0$. Then, for an arbitrary number $r \in (0,1)$ and ε sufficiently small, there exists a number $T(\varepsilon)$,*

$$T(\varepsilon) = \left[\frac{1}{a} \cdot \left([-mr \ln \phi(\varepsilon)]^{\frac{1}{2(3m-2)(2m-1)}} - b\right)\right]^{2(2m-1)(m-1)}, \tag{12}$$

where

$$\phi(\varepsilon) = \max\left\{\delta_0^{\frac{1}{m}}, \delta_i(\varepsilon), \gamma_i^2(\varepsilon), i = 1,2,3\right\}, \tag{13}$$

and a and b are some generic positive constants, so that

$$\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

P r o o f. By using the notations (9), we shall first estimate Δ_t^ε for t from any finite interval $[0,T]$. If we subtract the equations (3) and (4) and after that apply Ito differential formula to $(z_t^\varepsilon)^{2m}$, we find

$$(z_t^\varepsilon)^{2m} = (z_0^\varepsilon)^{2m} + 2mI_1(t) + m(2m-1)I_2(t) + 2mI_3(t),$$

where

$$I_1(t) = \int_0^t \Big[\tilde{a}_1(s,x_s^\varepsilon,\varepsilon) - a_1(s,x_s) + \int_0^s [\tilde{a}_2(s,r,x_r^\varepsilon,\varepsilon) - a_2(s,r,x_r)]\,dr + \int_0^s [\tilde{a}_3(s,r,x_r^\varepsilon,\varepsilon) - a_3(s,r,x_r)]\,dw_r\Big] \cdot (z_s^\varepsilon)^{2m-1}\,ds,$$

$$I_2(t) = \int_0^t \Big[\tilde{b}_1(s,x_s^\varepsilon,\varepsilon) - b_1(s,x_s) + \int_0^s [\tilde{b}_2(s,r,x_r^\varepsilon,\varepsilon) - b_2(s,r,x_r)]\,dr + \int_0^s [\tilde{b}_3(s,r,x_r^\varepsilon,\varepsilon) - b_3(s,r,x_r)]\,dw_r\Big]^2 \cdot (z_s^\varepsilon)^{2m-2}\,ds,$$

$$I_3(t) = \int_0^t \Big[\tilde{b}_1(s,x_s^\varepsilon,\varepsilon) - b_1(s,x_s) + \int_0^s [\tilde{b}_2(s,r,x_r^\varepsilon,\varepsilon) - b_2(s,r,x_r)]\,dr + \int_0^s [\tilde{b}_3(s,r,x_r^\varepsilon,\varepsilon) - b_3(s,r,x_r)]\,dw_r\Big] \cdot (z_s^\varepsilon)^{2m-1}\,dw_s.$$

Since $EI_3(t) = 0$ for every t, it follows that

$$\Delta_t^\varepsilon = \Delta_0^\varepsilon + 2mEI_1(t) + m(2m-1)EI_2(t), \quad t \in [0,T]. \tag{14}$$

The estimation of $EI_1(t)$ is based on Hölder inequality for $p = 2m$, $q = \frac{2m}{2m-1}$ and on the following well-known Ito formula (see [4, 5, 9, 10]): If f_t is any measurable $\mathcal{F}_t$-adapted process satisfying $\int_0^t Ef_s^{2m}\,ds < \infty$, then $E\left(\int_0^t f_s\,dw_s\right)^{2m} \le [m(2m-1)]^m t^{m-1} \int_0^t Ef_s^{2m}\,ds$. Then, we find

$$\begin{aligned} EI_1(t) \le \int_0^t \Big[& (E|a_1(s,x_s^\varepsilon) - a_1(s,x_s)|^{2m})^{\frac{1}{2m}} + (E|\alpha_1(s,x_s^\varepsilon,\varepsilon)|^{2m})^{\frac{1}{2m}} \\ & + s^{\frac{2m-1}{2m}} \Big(\int_0^s E|a_2(s,r,x_r^\varepsilon) - a_2(s,r,x_r)|^{2m}\,dr\Big)^{\frac{1}{2m}} \\ & + s^{\frac{2m-1}{2m}} \Big(\int_0^s E|\alpha_2(s,r,x_r^\varepsilon,\varepsilon)|^{2m}\,dr\Big)^{\frac{1}{2m}} \\ & + C\,s^{\frac{m-1}{2m}} \Big(\int_0^s E|a_3(s,r,x_r^\varepsilon) - a_3(s,r,x_r)|^{2m}\,dr\Big)^{\frac{1}{2m}} \\ & + C\,s^{\frac{m-1}{2m}} \Big(\int_0^s E|\alpha_3(s,r,x_r^\varepsilon,\varepsilon)|^{2m}\,dr\Big)^{\frac{1}{2m}}\Big] \cdot (\Delta_s^\varepsilon)^{\frac{2m-1}{2m}}\,ds, \end{aligned}$$

where $C = [m(2m-1)]^{1/2}$. By using the Lipschitz condition (2), the conditions (5) and (6), as well as the notation (13), we deduce that

$$EI_1(t) \le L \int_0^t \Delta_s^\varepsilon \, ds + \int_0^t \Big[\phi(\varepsilon) + L\, s^{\frac{2m-1}{2m}} \Big(\int_0^s \Delta_r^\varepsilon \, dr \Big)^{\frac{1}{2m}} + s\, \phi(\varepsilon) + CL\, s^{\frac{m-1}{2m}} \Big(\int_0^s \Delta_r^\varepsilon \, dr \Big)^{\frac{1}{2m}} + C\, \phi(\varepsilon)\, s^{\frac{m-1}{2m}} \Big] \cdot (\Delta_s^\varepsilon)^{\frac{2m-1}{2m}} \, ds.$$

By applying Young inequality $|a|^r \cdot |b|^{1-r} \le r|a| + (1-r)|b|$, $0 \le r \le 1$, and after that the integration by parts to the obtained double integrals, we finally get

$$EI_1(t) \le L \int_0^t \Big[1 + \frac{1+C}{2m}(t-s) + \frac{m-1}{m} s + \frac{C(2m-1)}{2m} s^{\frac{m-1}{2m}} \Big] \Delta_s^\varepsilon \, ds + \phi(\varepsilon) \int_0^t \Big[1 + s + s^{\frac{1}{2}} \Big] (\Delta_s^\varepsilon)^{\frac{2m-1}{2m}} \, ds. \tag{15}$$

Similarly, to estimate $EI_2(t)$, we apply Hölder inequality for $p = m$, $q = \frac{m}{m-1}$ and the preceding procedure. So, we have

$$\begin{aligned} EI_2(t) \le A \int_0^t \Big[& (E|b_1(s, x_s^\varepsilon) - b_1(s, x_s)|^{2m})^{\frac{1}{m}} + (E|\beta_1(s, x_s^\varepsilon, \varepsilon)|^{2m})^{\frac{1}{m}} \\ & + s^{\frac{2m-1}{m}} \Big(\int_0^s E|b_2(s, r, x_r^\varepsilon) - b_2(s, r, x_r)|^{2m} \, dr \Big)^{\frac{1}{m}} \\ & + s^{\frac{2m-1}{m}} \Big(\int_0^s E|\beta_2(s, r, x_r^\varepsilon, \varepsilon)|^{2m} \, dr \Big)^{\frac{1}{m}} \\ & + C^2\, s^{\frac{m-1}{m}} \Big(\int_0^s E|b_3(s, r, x_r^\varepsilon) - b_3(s, r, x_r)|^{2m} \, dr \Big)^{\frac{1}{m}} \\ & + C^2\, s^{\frac{m-1}{m}} \Big(\int_0^s E|\beta_3(s, r, x_r^\varepsilon, \varepsilon)|^{2m} \, dr \Big)^{\frac{1}{m}} \Big] \cdot (\Delta_s^\varepsilon)^{\frac{m-1}{m}} \, ds, \end{aligned}$$

where $A = 6^{\frac{2m-1}{m}}$. Finally, it follows that

$$EI_2(t) \le AL^2 \int_0^t \Big[1 + \frac{1+C^2}{m}(t-s) + \frac{m-1}{m} (s^{\frac{2m-1}{m-1}} + C^2 s) \Big] \Delta_s^\varepsilon \, ds + A\, \phi(\varepsilon) \int_0^t [1 + C^2 s + s^2] (\Delta_s^\varepsilon)^{\frac{m-1}{m}} \, ds. \tag{16}$$

By applying the elementary inequality $v^{r_2} \le v^{r_1} + v$ for any number $v \ge 0$ and $0 < r_1 \le r_2 < 1$, by setting $v = \Delta_t^\varepsilon$, $r_1 = (m-1)/m$, $r_2 = (2m-1)/(2m)$, we obtain $(\Delta_s^\varepsilon)^{\frac{2m-1}{2m}} \le (\Delta_s^\varepsilon)^{\frac{m-1}{m}} + \Delta_s^\varepsilon$. Now, since $0 \le t-s \le T$, the estimations (15) and (16) together with (14) imply that

$$\Delta_t^\varepsilon \le \phi^m(\varepsilon) + \int_0^t \Big[\eta_1 + \eta_2 T + \eta_3 s + \eta_4 s^{\frac{1}{2}} + \eta_5 s^{\frac{m-1}{2m-1}} + \eta_6 s^{\frac{2m-1}{m-1}}\Big]\Delta_s^\varepsilon \, ds$$

$$+\phi(\varepsilon)\int_0^t [\nu_1 + \nu_2 s + \nu_3 s^2 + \nu_4 s^{\frac{1}{2}}](\Delta_s^\varepsilon)^{\frac{m-1}{m}} ds, \tag{17}$$

where η_i and ν_i are some generic constants independent of T and ε. Now, it is convenient to apply the following generalized Gronwall–Bellman inequality [1]: Let $u(t)$, $a(t)$ and $b(t)$ be non-negative and continuous functions in $[0,T]$ and let $C > 0,\ 0 \le \gamma < 1$ be constants. If

$$u(t) \le C + \int_0^t a(s)u(s)\, ds + \int_0^t b(s)\, u^\gamma(s)\, ds, \quad t \in [0,T],$$

then $u(t) \le \left(C^{1-\gamma} e^{(1-\gamma)\int_0^t a(s)\,ds} + (1-\gamma)\int_0^t b(s)\, e^{(1-\gamma)\int_s^t a(r)\,dr} ds\right)^{\frac{1}{1-\gamma}}$.

Therefore, by putting $u(s) = \Delta_s^\varepsilon, \gamma = (m-1)/m$ to (17), we find

$$\Delta_t^\varepsilon \le \phi^m(\varepsilon)\cdot\theta(t)\cdot e^{\xi(t,T)}, \tag{18}$$

where

$$\theta(t) = (q_1 + q_2 t + q_3 t^2 + q_4 t^3 + q_5 t^{\frac{3}{2}})^m,$$
$$\xi(t,T) = p_1 t + r_1 Tt + r_2 t^2 + p_3 t^{\frac{3m-2}{m-1}} + p_4 t^{\frac{3m-2}{2m-1}} + p_5 t^{\frac{3}{2}}.$$

Clearly, $\sup_{[0,T]} \Delta_t^\varepsilon \to 0$ as $\varepsilon \to 0$.

Since the previous assertion does not hold for $t \in [0,\infty)$, we shall determine $T(\varepsilon)$ so that $\sup_{[0,T(\varepsilon)]} \Delta_t^\varepsilon \to 0$ as $\varepsilon \to 0$, by following the ideas from Section 2. Because the estimation (18) has the form as the one from paper [7], we give a short description for finding $T(\varepsilon)$.

First, there exists $\varepsilon_0 \in (0,1)$, so that $\phi(\varepsilon) \le \rho < 1$ for $\varepsilon \in (0,\varepsilon_0)$. From the requirement that $\phi^m(\varepsilon)\,\theta(T(\varepsilon))\,e^{\xi(T(\varepsilon),T(\varepsilon))} \to 0$ as $\varepsilon \to 0$, we determine $T(\varepsilon)$ from the relation

$$\xi(T(\varepsilon),T(\varepsilon)) \le -mr\ln\phi(\varepsilon), \quad \varepsilon \in (0,\varepsilon_0),$$

for an arbitrary number $r \in (0,1)$. The function ξ can be treated as a polynomial of the argument $u^{1/[(2(m-1)(2m-1)]}$, and, therefore, we find $T(\varepsilon)$ in the form (12) from the relation

$$\xi(T(\varepsilon),T(\varepsilon)) \le \Big[a\cdot(T(\varepsilon))^{\frac{1}{2(2m-1)(m-1)}} + b\Big]^{2(3m-2)(2m-1)} \le -mr\ln\phi(\varepsilon),$$

where a and b are some positive constants. They can be chosen so that

$$p_i \le C_{2(3m-2)(2m-1)}^{k_i} \cdot a^{k_i} \cdot b^{2(3m-2)(2m-1)-k_i},$$

for $k_1 = 2(2m-1)(m-1)$, $k_2 = 4(2m-1)(m-1)$ and $r_1 + r_2 = p_2$, $k_3 = 2(3m-2)(m-1)$, $k_4 = 2(3m-2)(2m-1)$, $k_5 = 3(2m-1)(m-1)$. Since the term with the largest power is $p_3 T_\varepsilon^{\frac{3m-2}{m-1}}$, the constant a is determined from the relation $p_3 = a^{2(3m-2)(2m-1)}$, while

$$b = \max_{i\in\{1,2,4,5\}} \left\{ \left[\frac{p_i}{C_{2(3m-2)(2m-1)}^{k_i} \cdot p_3^{1/[2(3m-2)(2m-1)]}} \right]^{1/[2(3m-2)(2m-1)-k_i]} \right\}.$$

Obviously, $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and

$$\sup_{t\in[0,T(\varepsilon)]} \Delta_t^\varepsilon \le (\phi(\varepsilon))^{m(1-r)} \theta(T(\varepsilon)) \equiv R(\varepsilon) \tag{19}$$

for $0 < \varepsilon \le \varepsilon_0 < 1$, and $R(\varepsilon) \to 0$ as $\varepsilon \to 0$, which completes the proof. ∎

Therefore, $R(\varepsilon)$ is a measure of the $(2m)$-th moment closeness of the solutions x^ε and x on the interval $[0, T(\varepsilon)]$. Since $\overline{R}(\varepsilon)$, given by (11) in Section 2, is a measure of the $(2m)$-th moment closeness on the interval $[0, \overline{T}(\varepsilon)]$, where $\overline{T}(\varepsilon)$ is expressed by (7), our main goal is to compare the lengths of these intervals, as well as the size of the closeness of the solutions x^ε and x.

Theorem 3. *If $\overline{T}(\varepsilon)$ and $T(\varepsilon)$ are given by* (7) *and* (12) *respectively, then the interval $[0, T(\varepsilon)]$ is much larger than $[0, \overline{T}(\varepsilon)]$, for ε sufficiently small.*

Proof. From (8) and (13) we observe that

$$(\phi(\varepsilon))^{2m} \le \overline{\phi}(\varepsilon), \quad \varepsilon \in (0, \min\{\overline{\varepsilon}_0, \varepsilon_0\}). \tag{20}$$

Now, from (7) and (12) it follows that

$$\begin{aligned} \frac{T(\varepsilon)}{\overline{T}(\varepsilon)} &\sim k \, \frac{(-\ln \phi(\varepsilon))^{(m-1)/(3m-2)}}{(-\ln \overline{\phi}(\varepsilon))^{1/(4m)}} \\ &\ge \frac{k}{(2m)^{1/(4m)}} \, (-\ln \phi(\varepsilon))^{(m-1)/(3m-2)-1/(4m)} \\ &\to \infty \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

where $k > 0$ is a constant. Therefore, $T(\varepsilon) \gg \overline{T}(\varepsilon)$ for ε sufficiently small. ∎

Remember that the both rates of the $(2m)$-th moment closeness between the solutions x^ε and x, obtained by using the method described in Section 2 and by applying Ito differential formula in Section 3, exponentially tend to zero as ε tends to zero. Since $[0,\overline{T}(\varepsilon)] \subset [0,T(\varepsilon)]$, it is convenient to compare these rates on the joint interval $[0,\overline{T}(\varepsilon)]$. First, let us note that from (18) we deduce

$$\sup_{[0,\overline{T}(\varepsilon)]} \Delta_t^\varepsilon \le \phi^m(\varepsilon)\cdot\theta(\overline{T}(\varepsilon))\cdot e^{\xi(\overline{T}(\varepsilon),\overline{T}(\varepsilon))} \equiv R_1(\varepsilon).$$

Corollary 1. *The rate of the $(2m)$-th moment closeness $R_1(\varepsilon)$ of the solutions x^ε and x, is much better than $\overline{R}(\varepsilon)$ on the interval $[0,\overline{T}(\varepsilon)]$, for ε sufficiently small.*

P r o o f. By using successively (19), the form of the polynomials P_4 and ξ, and after that (7), it follows that

$$\begin{aligned}\frac{\overline{R}(\varepsilon)}{R_1(\varepsilon)} &= \frac{(\overline{\phi}(\varepsilon))^{1-\overline{r}}\,P_4(\overline{T}^m(\varepsilon))}{(\phi(\varepsilon))^m\,e^{\xi(\overline{T}(\varepsilon),\overline{T}(\varepsilon))}\,\theta(\overline{T}(\varepsilon))}\\ &\ge \frac{(\overline{\phi}(\varepsilon))^{1-\overline{r}}}{(\overline{\phi}(\varepsilon))^{1/2}}\cdot\frac{P_4(\overline{T}^m(\varepsilon))}{\theta(\overline{T}(\varepsilon))\,e^{\xi(\overline{T}(\varepsilon),\overline{T}(\varepsilon))}}\\ &\ge k\,(\overline{\phi}(\varepsilon))^{1/2-\overline{r}}\cdot\frac{(\overline{T}(\varepsilon))^m}{(\overline{\phi}(\varepsilon))^{3m}}\cdot e^{-\xi(\overline{T}(\varepsilon),\overline{T}(\varepsilon))}\\ &\ge k\,(\overline{T}(\varepsilon))^{-2m}\cdot e^{-c(1/2-\overline{r})(\overline{T}(\varepsilon))^{4m}}\cdot e^{-a(\overline{T}(\varepsilon))^{(3m-2)/(m-1)}}\\ &\to\infty \quad\text{as}\quad \varepsilon\to 0,\quad \text{for}\quad \overline{r}>1/2\end{aligned}$$

and for some generic positive constants k,a,c. Therefore, $\overline{R}(\varepsilon) \gg R_1(\varepsilon)$ on the interval $[0,\overline{T}(\varepsilon)]$, for ε sufficiently small and an arbitrary $\overline{r}\in(1/2,1)$. ■

Finally, it is the most important to compare total rates of the $(2m)$-th closeness of the solutions x^ε and x on the intervals $[0,\overline{T}(\varepsilon)]$ and $[0,T(\varepsilon)]$.

Theorem 4. *The rate of the $(2m)$-th moment closeness $R(\varepsilon)$ of the solutions x^ε and x on the interval $[0,T(\varepsilon)]$ is much better than $\overline{R}(\varepsilon)$ on the interval $[0,\overline{T}(\varepsilon)]$, for ε sufficiently small.*

P r o o f. By using successively (20), the form of P_4 and θ, (7), (12), and again (20), we come to the following conclusion,

$$\frac{\overline{R}(\varepsilon)}{R(\varepsilon)} = \frac{(\overline{\phi}(\varepsilon))^{1-\overline{r}}\,P_4(\overline{T}^4(\varepsilon))}{(\phi(\varepsilon))^{m(1-r)}\,\theta(T(\varepsilon))}$$

$$
\begin{aligned}
&\geq c\,\frac{(\overline{\phi}(\varepsilon))^{2m(1-\overline{r})}}{(\overline{\phi}(\varepsilon))^{m(1-r)}}\cdot\frac{P_4(\overline{T}^m(\varepsilon))}{\theta(T(\varepsilon))}\\
&\geq c_1\,\frac{(\overline{T}(\varepsilon))^m}{(T(\varepsilon))^{3m}}\cdot(\overline{\phi}(\varepsilon))^{m(1+r-2\overline{r})}\\
&\sim c_2\,\frac{(-\ln\overline{\phi}(\varepsilon))^{1/4}}{(-\ln\overline{\phi}(\varepsilon))^{3m(m-1)/(3m-2)}}\cdot(\overline{\phi}(\varepsilon))^{m(1+r-2\overline{r})}\\
&= c_3\,\frac{(\overline{\phi}(\varepsilon))^{m(1+r-2\overline{r})}}{(-\ln\overline{\phi}(\varepsilon))^{3m(m-1)/(3m-2)-1/4}}\\
&\to\infty \quad\text{as}\quad \varepsilon\to 0,\quad \text{for}\quad \overline{r}>1/2,\ \text{and}\quad r<2\overline{r}-1,
\end{aligned}
$$

where c, c_1, c_2, c_3 are some generic positive constants. ■

Bearing in mind the previous facts, let us conclude that the estimations described in Section 3 improve the results from Section 2, because the rate of growing to infinity of the length of the interval $[0, T(\varepsilon)]$ is much larger than the one of $[0, \overline{T}(\varepsilon)]$, and the rate of the $(2m)$-th closeness $R(\varepsilon)$ of the solutions x^ε and x of the perturbed and unperturbed equations (4) and (3), is much better than $\overline{R}(\varepsilon)$. These facts are an essential reason to apply Ito differential formula in similar considerations, when the time-interval is infinite.

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