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A Modified Form of Nonlinear Differential Equations of Non-Newtonian Fluid Flows

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This paper deals with the solutions of nonlinear second order differential equations, arising in viscoelastic fluid flows in a rotating cylinder. Existence, uniqueness and analyticity results are obtained using the perturbation techniques and the Shauder theory. For special values of some parameters, we obtain some results given recently by Vajravelu et al. (2000).

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1. Introduction

In a recent paper, Vajravelu et al [6] have studied the existence, uniqueness, and behaviour of exact solutions of second order nonlinear differential equations arising in viscoelastic fluid flows in a rotating cylinder, and other problems [4]. Such problems were considered earlier by Garg and Rajagopal [2], Rajeswari and Rathna [5], Dunn and Rajagopal [3], and others.

In this paper, we consider a modified form of the steady equation [1] for the fluid in a cylinder of radius R and angular velocity Ω, in the following form:

$$\mu\left(\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}\right)+\beta\left(\frac{dv}{dr}-\frac{v}{r}\right)^2\left(b\frac{d^2v}{dr^2}-\frac{a}{r}\frac{dv}{dr}+\frac{av}{r^2}\right)=0,\ a,b>0, \tag{1}$$

with the boundary conditions

$$\begin{aligned} v &= R\,\Omega \quad \text{at} \quad r=R,\\ v &\to 0 \quad \text{as} \quad r\to\infty, \end{aligned} \tag{2}$$

where $v = v_\theta$ is the nonzero velocity in polar coordinates, and μ and β are material constants due to viscosity and viscoelasticity, respectively. Taking the dimensionless variables

$$\overline{r} = r/R \quad \text{and} \quad \overline{v} = \frac{v}{R\,\Omega}$$

and substituting in equation (1) and (2), we get (without bars),

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} + a\varepsilon\left(\frac{dv}{dr} - \frac{v}{r}\right)^2\left(\frac{b}{a}\frac{d^2v}{dr^2} - \frac{1}{r}\frac{dv}{dr} + \frac{v}{r^2}\right) = 0, \tag{3}$$

with the boundary conditions

$$\begin{aligned} v &= 1 \quad \text{at} \quad r = 1, \\ v &\to 0 \quad \text{as} \quad r \to \infty, \end{aligned} \tag{4}$$

and $\varepsilon = \frac{\Omega^2\beta}{\mu}$, the dimensionless number related to the material constants and the rotation of the cylinder.

2. Existence and uniqueness

Here our aim is to find $v = v\,(r)$ which satisfies (3) and (4). Let $A = \frac{dv}{dr} - \frac{v}{r}$, then equation (3), together with the boundary conditions given by (4), can be rewritten as

$$\left(1 + \varepsilon\ b\ A^2\right)\frac{d^2v}{dr^2} + \frac{A}{r}\left(1 - \varepsilon\ a\ A^2\right) = 0, \quad 1 < r < \infty, \tag{5}$$

$$v\,(1) = 1 \quad \text{and} \quad \lim_{r\to\infty}(r) = 0. \tag{6}$$

Dividing by $\left(1 + \varepsilon\ b\ A^2\right) > 0$, we get

$$\frac{d^2v}{dr^2} + \frac{1}{r}\left(\frac{1 - \varepsilon\ a\ A^2}{1 + \varepsilon\ b\ A^2}\right) A = 0. \tag{7}$$

Differentiating $A = \frac{dv}{dr} - \frac{v}{r}$ with respect to r, we obtain

$$\frac{dA}{dr} = \frac{d^2v}{dr^2} - \frac{1}{r}\frac{dv}{dr} + \frac{v}{r^2}, \tag{8}$$

$$\text{or} \quad \frac{d^2v}{dr^2} = \frac{dA}{dr} + \frac{A}{r}. \tag{9}$$

Substituting this into (7), we get

$$\frac{dA}{dr} + \frac{1}{r}\left(1 + \frac{1 - \varepsilon\ a\ A^2}{1 + \varepsilon\ b\ A^2}\right) A = 0, \quad A\,(1) = \lambda, \tag{10}$$

where λ is an unknown parameter. Let $\omega = \varepsilon\, A^2 > 0$, and consider the function

$$f\,(\omega) = \frac{1 - a\,\omega}{1 + b\,\omega} \tag{11}$$

subject to

$$\lim_{\omega \to 0} f\,(\omega) = 1 \quad \text{and} \quad \lim_{\omega \to \infty} f\,(\omega) = -\,\frac{a}{b}. \tag{12}$$

Then it follows that

$$f\,'(\omega) = -\,\frac{a + b}{(1 + b\,\omega)^2} < 0, \qquad \text{for} \quad a + b > 0 \tag{13}$$

and therefore, we have for $\omega > 0$:

$$-\frac{a}{b} < f\,(\omega) < 1 \;\; \text{and} \quad |f\,'(\omega)| < a + b. \tag{14}$$

Solving (10), we get

$$A\,(r) = \lambda\;\exp\left\{-\int_1^r \frac{1 + f\,\left(\varepsilon A^2\right)}{\eta}\; d\eta\right\}. \tag{15}$$

Observe that

$$\text{sign}\; A\,(r) = \text{sign}\;\lambda \qquad \text{and} \qquad |A\,(r)| < |\lambda|\,. \tag{16}$$

Moreover, we have the following

Proposition 1. *Recall* $A\,(r) = \lambda\;\exp\left\{-\int_1^r \frac{1+f\left(\varepsilon A^2\right)}{\eta}\; d\eta\right\}$ *and* $f\,(\omega) > -\frac{a}{b}$, *then for* $1 < r < \infty$, *we have*

$$|\lambda|\;\; r^{-2} \le\; |A\,(r)| \;\; \le |\lambda|\;\; r^{-\frac{b-a}{b}}, \tag{17}$$

$$|\lambda|\left|1 - \frac{a}{b}\right|\;\; r^{-3} \le\; |A'(r)| \;\; \le 2\,|\lambda|\;\; r^{-\frac{2b-a}{b}}. \tag{18}$$

P r o o f. First we see from (14) and (15) that

$$|A\,(r)| = |\lambda|\;\; e^{-\int_1^r \frac{1+f\left(\varepsilon A^2\right)}{\eta}\; d\eta} \le |\lambda|\;\; e^{-\int_1^r \frac{1-\frac{a}{b}}{\eta}\; d\eta} = |\lambda|\;\; r^{-\frac{b-a}{b}} \tag{19}$$

and

$$|A'(r)| = \frac{1}{r}\;\left|1 + f\,\left(\varepsilon A^2\right)\right| |A| \ge \frac{1}{r}\;|\lambda|\;\;\left|1 - \frac{a}{b}\right|\;\; r^{-2} = |\lambda|\left|1 - \frac{a}{b}\right|\;\; r^{-3}, \tag{20}$$

furthermore, since $f(\omega) < 1$, then we have

$$|A(r)| \geq |\lambda| \quad e^{-2(\ln r - \ln 1)} = |\lambda| \quad r^{-2} \tag{21}$$

and

$$|A'(r)| = \frac{1}{r} \left|1 + f\left(\varepsilon A^2\right)\right| |A| \ \leq \frac{2}{r} \ |\lambda| \quad r^{-\frac{b-a}{b}} = 2\,|\lambda| \quad r^{-\frac{2b-a}{b}}. \tag{22}$$

Combining these inequalities we get the required results. ■

From (22), we see that

$$|A'(r)| \leq 2\,|\lambda| \quad r^{-\frac{2b-a}{b}} \leq 2\,|\lambda| \ , \qquad 1 \leq r < \infty \tag{23}$$

which implies a uniform continuity on $1 \leq r < \infty$.

Next we define the mapping $T : B \to B$, where B is the Banach space of bounded continuous functions on $1 \leq r < \infty$ with norm $\|g\| = \sup_{1 \leq r < \infty} |g(r)|$ for all $g \in B$, via the formula

$$(Tg)(r) = \lambda \ \exp\left\{-\int_1^r \frac{1 + f\left(\varepsilon A^2\right)}{\eta} \ d\eta\right\}, \qquad 1 \leq r < \infty. \tag{24}$$

From the analysis for (19), we see that

$$|(Tg)(r)| \leq |\lambda| \quad r^{-\frac{b-a}{b}}, \qquad 1 \leq r < \infty. \tag{25}$$

Differentiating Tg , since $(Tg)(r)$ is a continuously differentiable function on $1 \leq r < \infty$ with norm $\|Tg\| < |\lambda| < \infty$, we obtain

$$(Tg)'(r) = -\frac{1 + f\left(\varepsilon \ g^2(r)\right)}{r} \ (Tg)(r)\,. \tag{26}$$

From the analysis for (22), we see that

$$|(Tg)'(r)| \leq 2\,|\lambda| \quad r^{-\frac{2b-a}{b}}, \qquad 1 \leq r < \infty. \tag{27}$$

Therefore, we conclude that the image TB consists of functions bounded by (24) and equicontinuous by (27).

Now we define a subspace S_λ of B as follows

$$\begin{aligned} S_\lambda \ = \ \Big\{ & g \in B : |g(r)| \leq |\lambda| \quad r^{-\frac{b-a}{b}} \ , 1 \leq r < \infty, \text{ and} \\ & |g(r_1) - g(r_2)| \leq 2\,|\lambda|\,|r_1 - r_2| \ , 1 \leq r_1 \leq r_2 < \infty\Big\}\,. \end{aligned} \tag{28}$$

Consequently, using (24), we see that $TB \subset S_\lambda$.

Next we state and prove the existence and the uniqueness theorem, which is the main result of this paper.

Theorem 2. *For each* $\lambda \in (-\infty, \infty)$, *there exists a unique function* $A_\lambda \in S_\lambda$, *which depends continuously upon* λ, *such that*

$$A_\lambda(r) = \lambda \ \exp\left\{-\int_1^r \frac{1+f\left(\varepsilon A_k^2(\eta)\right)}{\eta}\, d\eta\right\}, \tag{29}$$

where

$$\left|A_\lambda'(r)\right| \le 2|\lambda| \ r^{-\frac{2b-a}{b}}. \tag{30}$$

A proof of this fundamental theorem is presented in the next three propositions. We begin first by showing the continuity of the map $T : S_\lambda \to S_\lambda$.

Proposition 3. *Recall* $(Tg)(r) = \lambda \ \exp\left\{-\int_1^r \frac{1+f(\varepsilon A^2)}{\eta}\, d\eta\right\}$, *then we have*

$$\|Tg_1 - Tg_2\| \le |\lambda|^2 \ (a+b) \ b\,\varepsilon\, \|g_1 - g_2\| \,. \tag{31}$$

P r o o f. Using the mean value theorem twice, we get

$$\begin{aligned} Tg_1(r) - Tg_2(r) &= \lambda\left\{ e^{-\int_1^r \frac{1+f\left(\varepsilon g_1^2\right)}{\eta}\, d\eta} - e^{-\int_1^r \frac{1+f\left(\varepsilon g_2^2\right)}{\eta}\, d\eta}\right\} \\ &= -\lambda e^{-\delta}\int_1^r \frac{f\left(\varepsilon g_1^2\right) - f\left(\varepsilon g_2^2\right)}{\eta}\, d\eta \\ &= -\lambda e^{-\delta}\int_1^r \frac{\varepsilon\, f'(c_\eta)\left(g_1^2 - g_2^2\right)}{\eta}\, d\eta \end{aligned}$$

and therefore, for $1 \le r < \infty$, we have

$$\begin{aligned} |Tg_1(r) - Tg_2(r)| &\le |\lambda| \ \|g_1 - g_2\| \int_1^r \frac{\varepsilon\, f'(c_\eta) \ |g_1 + g_2|}{\eta}\, d\eta \\ &\le |\lambda| \ (a+b)\,\varepsilon \int_1^r \frac{|\lambda| \ \eta^{-\frac{b-a}{b}}}{\eta}\, d\eta \times \|g_1 - g_2\| \\ &\le |\lambda|^2 \ b\,\varepsilon\, \frac{a+b}{a-b}\left(1 - r^{-\frac{b-a}{b}}\right) \times \|g_1 - g_2\| \\ &\le |\lambda|^2 \ (a+b) \ b\,\varepsilon\, \|g_1 - g_2\| \,. \end{aligned} \tag{32}$$

Taking the supremum over $r \in [1, \infty)$, we get the required result. ■

From the above proposition, we have shown that T is a continuous map of S_λ into S_λ. Therefore by the Shauder theorem there exists a fixed point of T, and hence we proved the existence part. For the uniqueness part, we need to show first the following proposition.

Proposition 4. *A_λ is continuous with respect to λ.*

Proof. Let $A_{\lambda_1}(r)$ and $A_{\lambda_2}(r)$ be solutions that satisfy (31), then using the mean value theorem twice, we have

$$\begin{aligned} A_{\lambda_1}(r) - A_{\lambda_2}(r) &= (\lambda_1 - \lambda_2)\ e^{-\int_1^r \frac{1+f\left(\varepsilon A^2_{\lambda_1}(\eta)\right)}{\eta}\, d\eta} \\ &\quad + \lambda_2 \left[e^{-\int_1^r \frac{1+f\left(\varepsilon A^2_{\lambda_1}(\eta)\right)}{\eta}\, d\eta} - e^{-\int_1^r \frac{1+f\left(\varepsilon A^2_{\lambda_2}(\eta)\right)}{\eta}\, d\eta} \right] \\ &= (\lambda_1 - \lambda_2)\ e^{-\int_1^r \frac{1+f\left(\varepsilon A^2_{\lambda_1}(\eta)\right)}{\eta}\, d\eta} \end{aligned}$$

$$-\lambda_2\ \varepsilon\ e^{-\delta} \int_1^r \frac{f\,'(c_\eta)}{\eta} \left(A_{\lambda_1}(\eta) - A_{\lambda_2}(\eta)\right)\left(A_{\lambda_1}(\eta) + A_{\lambda_2}(\eta)\right) d\eta.$$

Taking absolute value of both sides together with (16), we get

$$|A_{\lambda_1}(r) - A_{\lambda_2}(r)|$$

$$\leq |\lambda_1 - \lambda_2| + \varepsilon\ (a+b)\,|\lambda_2|\ \left(|\lambda_1| + |\lambda_2|\right) \int_1^r |A_{\lambda_1}(\eta) - A_{\lambda_2}(\eta)|\, d\eta. \tag{33}$$

Applying the Gronwall inequality, we get

$$|A_{\lambda_1}(r) - A_{\lambda_2}(r)| \leq |\lambda_1 - \lambda_2| \left[e^{(r-1)k} + \frac{1}{k}\left(e^{(r-1)k} - 1\right)\right], \tag{34}$$

$$\text{with } k = \varepsilon\ (a+b)\,|\lambda_2|\ \left(|\lambda_1| + |\lambda_2|\right), \tag{35}$$

which proves our claim. ■

Proposition 5. *The solution $A_\lambda(r)$ is unique.*

Proof. Suppose there is another solution $B_\lambda(r)$ that satisfies (32) for $\lambda\,(=\lambda_1 = \lambda_2)$, then (33) implies that $B_\lambda(r) = A_\lambda(r)$. ■

By this we end the proof of our theorem. Now we will give a representation for v. Recall by definition that

$$\frac{dv}{dr} - \frac{1}{r}\, v = A_\lambda(r), \quad 1 < r < \infty, \tag{36}$$

$$v(1) = 1 \quad \text{and} \quad \lim_{r\to\infty} v(r) = 0, \tag{37}$$

whose solution is

$$v(r) = r + r \int_1^r \frac{A_\lambda(\delta)}{\delta}\, d\delta, \tag{38}$$

or equivalently:

$$v(r) = r + \lambda r \int_1^r \frac{e^{-\int_1^\delta \frac{1+f\left(\varepsilon\ A_\lambda^2(\eta)\right)}{\eta}\, d\eta}}{\delta}\, d\delta. \tag{39}$$

From the second condition of (37) λ needs to be negative, i.e. $\lambda = -\alpha$, $\alpha > 0$, thus (39) becomes

$$v(r) = r - \alpha r \int_1^r \frac{e^{-\int_1^\delta \frac{1+f\left(\varepsilon\ A_{-\alpha}^2(\eta)\right)}{\eta}\, d\eta}}{\delta}\, d\delta. \tag{40}$$

Considering the case $v(R) = 0$ for some $R > 1$, then α must satisfy

$$\alpha = \frac{1}{I}\ , \quad \text{with} \quad I \triangleq \int_1^R \frac{e^{-\int_1^\delta \frac{1+f\left(\varepsilon\ A_{-\alpha}^2(\eta)\right)}{\eta}\, d\eta}}{\delta}\, d\delta. \tag{41}$$

Now since $-\frac{a}{b} < f(\omega) < 1$, then we get

$$\frac{1-R^{-2}}{2} \le I \ \le \frac{b}{b-a}\left(1 - R^{-\frac{b-a}{b}}\right), \tag{42}$$

and therefore we have

$$\frac{b-a}{b\left(1 - R^{-\frac{b-a}{b}}\right)} \le \alpha \ \le \frac{2}{1-R^{-2}}. \tag{43}$$

Letting $R \to \infty$ in (43), we get

$$\frac{b-a}{b} \le \alpha \le 2. \tag{44}$$

3. Perturbation analysis and solution

In this section we obtain the exact and the approximate solution to

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} + \varepsilon\left(\frac{dv}{dr} - \frac{v}{r}\right)^2\left(b\frac{d^2v}{dr^2} - \frac{a}{r}\left(\frac{dv}{dr} - \frac{v}{r}\right)\right) = 0 \tag{45}$$

$$v(1)=1, \qquad v(\infty)=0, \qquad 1<r<\infty\ . \tag{46}$$

First we begin by finding the approximate solution for every ε. For small $\varepsilon \ll 1$, we let $v = v_0 + \varepsilon v_1 + O\left(\varepsilon^2\right)$ in (45) and obtain the $O(1)$ problem

$$\frac{d^2 v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr} - \frac{v_0}{r^2} = 0, \qquad v_0(1) = 1, \qquad v_0(\infty) = 0 \tag{47}$$

and the $O(\varepsilon)$ problem

$$\begin{aligned} \frac{d^2 v_1}{dr^2} + \frac{1}{r}\frac{dv_1}{dr} - \frac{v_1}{r^2} &= -\left(\frac{dv_0}{dr} - \frac{v_0}{r}\right)^2 \left(b\,\frac{d^2 v_0}{dr^2} - \frac{a}{r}\left(\frac{dv_0}{dr} - \frac{v_0}{r}\right)\right), \\ v_1(1) &= 1, \qquad v_1(\infty) = 0\ . \end{aligned} \tag{48}$$

Solving (47) gives $v_0 = \frac{1}{r}$, substituting this in (48) yields

$$\begin{gathered} \frac{d^2 v_1}{dr^2} + \frac{1}{r}\frac{dv_1}{dr} - \frac{v_1}{r^2} = -\frac{8\,(a+b)}{r^5}, \\ v_1(1) = 1, \qquad v_1(\infty) = 0, \end{gathered} \tag{49}$$

whose solution is given by

$$v_1 = \frac{(a+b)}{3} \times \left(\frac{1}{r} - \frac{1}{r^5}\right). \tag{50}$$

Therefore the solution for the problem (45) for small ε is

$$v = \frac{1}{r} + \frac{(a+b)\,\varepsilon}{3} \times \left(\frac{1}{r} - \frac{1}{r^5}\right) + O\left(\varepsilon^2\right). \tag{51}$$

Now if ε is large then we let $\eta = \frac{1}{\varepsilon}$ in (45) to obtain

$$\eta\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right) + \left(\frac{dv}{dr} - \frac{v}{r}\right)^2 \left(b\frac{d^2 v}{dr^2} - \frac{a}{r}\left(\frac{dv}{dr} - \frac{v}{r}\right)\right) = 0 \tag{52}$$

$$v(1)=1, \qquad v(\infty)=0, \qquad 1<r<\infty\ . \tag{53}$$

Let $v = v_0 + \eta v_1 + O\left(\eta^2\right)$; then either $\frac{dv_0}{dr} - \frac{v_0}{r} = 0$ whose solution is given by $v_0 = cr$, or $b\ \frac{d^2 v_0}{dr^2} - \frac{a}{r}\left(\frac{dv_0}{dr} - \frac{v_0}{r}\right) = 0$ which gives $v_0 = c_1 r + c_2\, r^{\frac{a}{b}}$. In either case it is not possible to satisfy the infinity condition. Now since we failed to obtain an approximate solution for the problem (45) when ε is large, we have to solve it exactly. Let $w = \frac{\sqrt{\varepsilon}}{r}\,v$, then the problem (45) becomes

$$r\,\frac{d^2 w}{dr^2} + 3\frac{dw}{dr} + r^2\left(\frac{dw}{dr}\right)^2\left(br\,\frac{d^2 w}{dr^2} + (2b-a)\,\frac{dw}{dr}\right) = 0, \tag{54}$$

$$w(1) = \sqrt{\varepsilon}, \qquad w(\infty) = 0. \tag{55}$$

Moreover, letting $\tau = \ln r$, we get

$$\frac{d^2w}{d\tau^2} + 2\frac{dw}{d\tau} + \left(\frac{dw}{d\tau}\right)^2 \left(b\frac{d^2w}{d\tau^2} + (b-a)\frac{dw}{d\tau}\right) = 0, \tag{56}$$

$$w(0) = \sqrt{\varepsilon}, \qquad w(\infty) = 0, \tag{57}$$

and integrating equation (56) with respect to τ, gives

$$\frac{dw}{d\tau} \left[1 + \frac{b-a}{2}\left(\frac{dw}{d\tau}\right)^2\right]^{\frac{a+b}{2(b-a)}} - ce^{-2\tau} = 0. \tag{58}$$

If we take $b = 3a$ as a special case of equation (58), then we get

$$\left(\frac{dw}{d\tau}\right)^3 + \frac{1}{a}\frac{dw}{d\tau} - ce^{-2\tau} = 0, \tag{59}$$

whose solution is given in [6] as

$$\frac{dw}{d\tau} = \frac{\left[\sqrt[3]{g(\tau)} - g(\tau)\right]^{-\frac{1}{3}}}{6} \equiv G(\tau, c), \tag{60}$$

where

$$g(\tau) = 108\ ce^{-2\tau} + 6\sqrt{6 + 324\ c^2e^{-4\tau}}. \tag{61}$$

Integrating equation (60) with respect to τ, gives

$$w(\tau) = -\int_{\tau}^{\infty} G(p, c)\ dp. \tag{62}$$

Using the boundary condition $w(0) = \sqrt{\varepsilon}$, we find an implicit equation for the constant c

$$\sqrt{\varepsilon} = -\int_{0}^{\infty} G(p, c)\ dp \tag{63}$$

and therefore, the exact solution for the problem (45) is given by

$$v(\tau) = \frac{-r}{\sqrt{\varepsilon}} \int_{\ln r}^{\infty} G(p, c)\ dp \tag{64}$$

Now since $G(\tau, c) \sim 2ce^{-2\tau}$ for $\tau \to \infty$, the asymptotic behavior of $v(r)$ is given by

$$v(r) \sim \frac{-c(\varepsilon)}{r\sqrt{\varepsilon}} \quad \text{as } r \to \infty. \tag{65}$$

For special values of the parameters a and b, the results of Vajravelu et al. [6] are recovered.

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