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Physical Consistency of Fractional ΨDO

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In previous papers we have investigated a fractional calculus via pseudo–differential operators and shown its efficiency along accompanying experiments. The approach, briefly described in this paper, was at first based on Fourier transforms on the space $\mathbf{L}_2$ of square–summable functions. Unfortunately, it delivers "causal" solutions only for stable systems, as demonstrated below. This problem can be overcome by extending the approach to $\mathcal{D}'$, such that also unstable cases are modelled "physically consistent". But there is more influence of the mathematical approach on physically important properties. E.g., due to the non–local nature of fractional operators, a mathematical model has to include the whole past of the system. This cannot be done via local initial conditions. Different from many others, our approach includes the past by its nature. Last, not least we discuss the role of branches and branch cuts in the definition of fractional terms in differential equations with regard to real, causal and stable solutions.

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1. Preliminaries

1.1. Notations

1. We denote the imaginary unit by i and real– and imaginary parts by $\Re(s)$, $\Im(s)$.

2. We write $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$, $\forall$ denotes "for all" and $\theta(t)$ the unitstep.

3. We abbreviate pseudo-differential operator by ΨDO.

4. We write $\mathbf{L}_2$ for the set of square-summable functions $\mathbb{R} \to \mathbb{C}$ as well as for the vector space of its equivalence classes. Consequently, we write $g(t) = f(t)$ without "a.e.".

5. The space $\mathbf{L}_{\mathrm{loc}}$ is the space of locally integrable functions $\mathbb{R} \to \mathbb{C}$.

6. All non-integer powers s^q ($s \in \mathbb{C}$) are defined as principal branches with cut $\mathbb{R}_0^-$, i.e., in accordance with computer systems like *Mathematica, Maple, Mathlab*: $-\pi < \arg(s) \leq \pi$.

1.2. Distributions

$\mathcal{D}$ denotes the test space of C^∞-functions ($\mathbb{R} \to \mathbb{C}$) with compact support. The dual space $\mathcal{D}'$ of continuous linear functionals $f : \varphi \mapsto \langle f, \varphi \rangle$, $\varphi \in \mathcal{D}$ is called "space of distributions". Operations and properties in $\mathcal{D}'$ are induced from $\mathcal{D}$. For applications those definitions are not very helpful. All the more are the following basic properties (for more see e.g. [9]).

Properties:

1. The space $\mathbf{L}_{\mathrm{loc}} \subset \mathcal{D}'$ can be identified with the space of *regular* distributions:

$$\text{If } f \in \mathbf{L}_{\mathrm{loc}}, \quad \text{then} \quad f \overset{\text{id.}}{:=} \langle f, \varphi \rangle := \int_{-\infty}^{\infty} f(t)\,\varphi(t)\,dt \quad \forall\, \varphi \in \mathcal{D}.$$

2. Any element $f \in \mathcal{D}'$ is $\mathcal{D}'$–limit of $\mathbf{L}_{\mathbf{loc}}$–sequences.

3. Every $\mathcal{D}'$–element has unique derivatives of arbitrary order which fulfil the common rules. Vice versa every $\mathcal{D}'$–element has primitives which only differ by constants.

4. In general, the approach to products in $\mathcal{D}'$ is very expensive (see e.g. [8]). But, sufficient for most applications, one can use property 2 to embed pointwise $\mathbf{L}_{\mathbf{loc}}$–products and $\mathcal{D}'$–limits of according sequences into $\mathcal{D}'$.

1.3. Fourier transformations

1. We start from the Fourier transform $\widehat{x}(\omega)$ of some $\mathbf{L_2}$–function $x(t)$

$$\widehat{x}(\omega) := \mathcal{F}\{x(t)\} := \frac{1}{\sqrt{2\pi}}\,\mathbf{p.v.}\int_{-\infty}^{\infty} x(t)\,e^{-i\omega t}dt\,. \tag{1}$$

The principle value definition of the integral (**p.v.**) is needed to the extend the scope from $\mathbf{L_1} \cap \mathbf{L_2}$ to $\mathbf{L_2}$. This way $\mathcal{F}$ is a unitary isomorphism on $\mathbf{L_2}$. (e.g. [9,11])

2. For the sophisticated procedure to extend the Fourier transformation on distributional spaces we refer to [5, 8, 12]. Finally, the Fourier transform of any $\mathcal{D}'$ element f is together with its inverse well–defined via

$$\mathcal{F}\{f\} := \left\langle \widehat{f},\, \varphi \right\rangle := \langle f,\, \widehat{\varphi} \rangle \quad \forall\, \varphi \in \mathcal{F}(\mathcal{D})\,,$$

such that $\widehat{\varphi} \in \mathcal{D}$. (For the characterization of $\mathcal{F}(\mathcal{D})$ see e.g. [2, 6, 9]).

3. As already claimed above in the beginning of 1.2, this definition is not very helpful to calculate explicitly some $\widehat{f}$ (see the following remark). But it is very important that this definition preserves by nature all rules of Fourier transforms.

Remark 1. Even the scope of integral (1) is much larger then $\mathbf{L_2}$. Furthermore, one can generate from any known Fourier transform chains of $\mathcal{D}'$–correspondencies (e.g., via the differentiation laws).

2. Fractional calculus via ΨDO

2.1. The functional calculus definition in $\mathcal{D}'$

Applying $\mathcal{F}^{-1}$ to the differentiation law $\mathcal{F}\{D^n x(t)\} = (i\omega)^n \mathcal{F}\{x(t)\}$ one can identify the operator D^n with $\mathcal{F}^{-1}(i\omega)^n \mathcal{F}$. Together with its linearity $\mathcal{F}$ provides this way a technique to define ΨDOs replacing $(i\omega)^n$ by a convenient $G(i\omega)$, called *symbol*. Under the weak condition that $G\widehat{x}(t) \in \mathcal{D}'$, a well–defined ΨDO is established via

$$\mathcal{G} := G(D) := \mathcal{F}^{-1}\, G(i\omega)\; \mathcal{F}.$$

From Fourier analysis the behaviour of the operator and associated solutions can be discussed along the spectrum (zero–set) of the symbol G. This procedure is called *functional calculus* (see e.g. [3, 8]). We will now use it for a fractional calculus.

Definition. (Fractional ΨDO) Let $q \in \mathbb{R}_0^+$.

1. Define all fractional powers $(i\omega)^{q_k}$ of the symbol as principal branches,

$$(i\omega)^{q_k} := |\omega|^{q_k} \exp\left(i \operatorname{sign}(\omega)\, q_k \pi/2\right).$$

2. Linear fractional ΨDOs $A(D) = \sum_{k=0}^{n} a_k D^{q_k}$, $a_k \in \mathbb{R}$ are now defined via

$$\mathcal{A} := A(D) = \mathcal{F}^{-1}\, A(i\omega)\; \mathcal{F}.$$

3. Consequently, a *fractional derivative* of order q is given by

$$D^q f := \mathcal{F}^{-1}\; (i\omega)^q\; \widehat{f}.$$

4. The related function $A : s \mapsto A(s)$ is called the *symbol* of the ΨDO. The kernel of $\mathcal{A}$ is

$$\ker(\mathcal{A}) := \{\, x(t) \mid \mathcal{A}x(t) = 0 \}. \tag{2}$$

Remark 2.

1. From experience, this definition covers all physically relevant cases at least in mechanics. For a further discussion of the scope of such definitions see e.g. [10].

2. Different from many other definitions the case $q \in \mathbb{N}_0$ is included automatically. The same holds for the semigroup property which is obviously inherited from the symbols:

$$D^q D^p = D^p D^q = D^{p+q}.$$

3. The entire symbol $A(s)$ characterizes totally the ΨDO. Over all, its zero–set is important.

4. Easy is the inversion of an operator $\mathcal{A}$ (in case of existence):

$$\mathcal{A}^{-1} := \mathcal{F}^{-1}\ (1/A(i\omega))\ \mathcal{F}\,. \tag{3}$$

5. We dispense with a description of fractional calculi via a priori definitions of fractional derivatives (see e.g. [7, 10]). We have shown in [5] that Riemann–Liouville– as well as Caputo–integrals with lower bonds $-\infty$ are included in the ΨDO–approach.

With respect to Definition , 4., we cite a very important and a little bit surprising fact about linear ΨDOs with constant coefficients (for a proof see [5]):

Theorem. *The kernel (2) of $\mathcal{A}$ is just the same as the one of an integer ordered operator with the same zero–set. More precisely:*

Let $A\,(s)$ denote the symbol of $\mathcal{A}$ and let its zeros be $s_k = \sigma_k \pm i\nu_k$ with multiplicity m_k.

Then the real solutions of $\mathcal{A}\,x(t) = 0$ are given by all $x(t\,)$:

$$x(t\,) = \sum_k e^{\sigma_k t}\Big(p_{m_k}(t\,)\sin(\nu_k t) + q_{m_k}(t\,)\,\cos(\nu_k t)\Big), \tag{4}$$

where p_{m_k}, q_{m_k} are arbitrary real polynomials of degree $\leq m_k - 1$.

3. Physical consistency

We will focus on qualitative criteria, i.e. roughly, a mathematical model of a physical system should not deliver results that contradict physical laws.

Let us discuss some problems along a mathematical model of shape

$$\mathcal{A}\,x(t) = f(t), \qquad \mathcal{A} := A(D) \quad \text{(see Definition , 3.)}\,. \tag{5}$$

With respect to (5), the output $x(t)$ depends on the input $f(t)$ via $x(t) = \mathcal{A}^{-1} f(t)$. Since $\mathcal{A}$ is linear, one can represent $x(t)$ via a convolution with the so–called *impulse response* $k(t)$:

$$x(t) = (f * k)\,(t) := \int_{-\infty}^{\infty} k(u)\, f(t-u)\, du\,, \quad \text{where } k \text{ solves } \mathcal{A}\,k(t) = \delta(t). \tag{6}$$

Note that the past of the system, determined by $f(t)$, $t \leq 0$, is included automatically. Thus we can concentrate our considerations on impulse responses. As written down in several papers (e.g. [1, 5]) we arrive via (3) at

$$k(t) = \mathcal{F}^{-1}\left(1/A(i\omega)\right) \widehat{\delta}(\omega) \overset{i\omega \mapsto s}{=} \frac{1}{2\pi i}\, \mathbf{p.v.} \int_{-i\infty}^{i\infty} \frac{e^{st}}{A(s)}\, ds. \tag{7}$$

3.1. The role of the timescale

Aware, that this is an endless field in and outside "exact" sciences, we will keep very short.

Above all, we will demand reproducibility of physical laws. Hence, any model should be translation invariant with respect to time. Concerning fractional operators it is

$$\mathcal{A}\,\{x(t)\}|_{t=t_0} = \mathcal{A}\,\{x(t+t_0)\}|_{t=0} \qquad \text{for all fixed } \; t_0 \in \mathbb{R}.$$

Obviously, this is true for our ΨDOs in terms of Definition. Moreover, the past is included via convolution (6).

Now, let us compare with approaches based on one–sided Laplace–transforms. We have shown that they are by nature not time–invariant ([4]). Truly, there arises no problem, if the origin is set as long as the system is totally in rest. Even the same impulse response (7) is received, if (and only if) all "initial conditions" vanish. But what, if a past must be taken into account ?

In case of differential equations with only integer derivatives one commonly models the past via a finite set of initial conditions. This is ensured by the local character of integer ordered differentiation for a wide class of operators. In case of fractional, i.e. global operators, this is not possible ([4]). Any change of the input $f(t)$ for $t \leq 0$ changes the output $x(t)$ for $t > 0$.

We conclude that approaches to fractional models which use one–sided Laplace transforms are not physically consistent. With some care (time transforms may cause false results) they may be used for special classes of problems. But, they are not at all appropriate for modelling, e.g. steady–state oscillations.

3.2. The role of function spaces

With respect to time, the output of some model should not depend on the future of the system. Concerning our model this is exactly the case, if the impulse response is causal, i.e.:

The model (5) is causal, if and only if $k(t) = 0 \quad \forall\, t \leq 0\,.$

We claim that the verification of this criterion depends on the choice of a sufficiently large function space. For demonstration we consider the model of a viscous damped single–mass–oscillator (see e.g. [1, 5])

$$\left(D^2 + aD^q + b\right)\, x(t) = f(t), \quad 0 < q < 2, \;\; a, b \neq 0.$$

The symbol of the inverse operator is $1/A(s) = 1/\left(s^2 + as^q + b\right)$, which is $(b \neq 0)$ obviously a $\mathbf{L_2}$–function. Consequently, equation (7) delivers a $\mathbf{L_2}$–solution too. An analysis, firstly given in [1], shows that this solution is causal, if and only if the symbol $s^2 + as^q + b$ has no zeros with positive real parts. This is known as (asymptotically) stable case and thus this model is at the first glance only physical consistent for stable problems. But let us discuss the unstable case along the explicit example $A(D) = D^2 - 0.2\, D^{0.1} + 0.2$ (the zeros of the symbol are $0.057687 \pm 1.97672\, i$). The $\mathbf{L_2}$–approach delivers consequently a (stable) $\mathbf{L_2}$–impulse response too, depicted in the left hand part of Figure 1.

Figure 1. Non–causal and causal impulse responses to
$\left(D^2 - 0.2\, D^{0.1} + 0.2\right)\, x(t) = \delta(t).$

Of course this solution is physical nonsense, due to its non–causality. Additionally, it is stable. But the composition of this $k(t)$ is such that the part

for $t \leq 0$ is just a kernel function (essentially not square–summable). Hence, via superposition, we get the physically consistent, unstable impulse response, depicted as right hand part in Figure 1. This demonstrates drastically that we had to extend our approach to a sufficiently large space to generate causal stable and unstable solutions. This is, why we embedded our operators in $\mathcal{D}'$.

3.3. Fractional powers and branch cut

First, we refer to a previous result in [1] that all fractional powers of the symbol must be defined by their principal branches to ensure causality as well as the continuity of solutions. All fractional powers must have the same domain on the logarithmic surface, such that it covers $\mathbb{C}$ and contains the real axis. This is remarkable, because the functional–calculus–definition requires only that the symbol has to be measurable. Thus, the principal–branch–definition has only physical reasons.

But what about the branch cut ? The common choice is $\mathbb{R}_0^-$. Figure 2 shows the according paths of integration we have used to get the above described physically consistent impulse responses.
As a simple reference example we consider now the impulse response of the model of a fractional differentiator

$$D^q k(t) = \delta(t) \quad (0 < q < 1).$$

The gross–formula (7) delivers

$$k(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{s^q}\, ds. \tag{8}$$

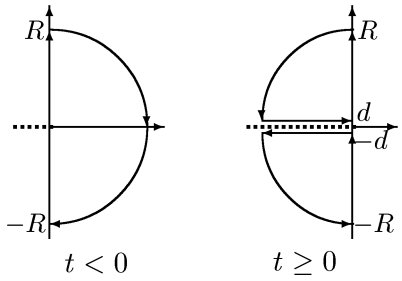

Figure 2: Integration paths.

The integrals along all arcs vanish for $R \to \infty$. Thus the only contribution to $k(t)$ is given for $t > 0$ by the two integrals along the branch cut for $d \to 0$. The weak singularity $s = 0$ does not matter, and we arrive at the physically consistent, causal result

$$k(t) = \frac{\theta(t) t^{q-1}}{\Gamma(q)}\,. \tag{9}$$

Remark 3. For $q = 1$ residue theorem delivers $k(t) = \theta(t)$. For all other $n + q$, $n \in \mathbb{N}$, $0 \leq q < 1$, the substitution $k_n(t) := D^n k(t)$ causes a weak integral–kernel in formula (8). Thus, via residue theorem again, equation (9) is valid for all $q \in \mathbb{R}^+$.

Consider now the general case, i.e., the cut is fixed by an argument $\phi : 0 \leq \phi < 2\pi$. Apart from causal or non–causal results, the non–vanishing parts, i.e. the integrals along the branch cut, always deliver functions of shape

$$C(q,\phi)\, t^{q-1} \quad \text{with} \quad C(q,\phi) \notin \mathbb{R}, \ \text{ if } \ \phi \neq \pi, \ \ q \notin \mathbb{N}.$$

But, the kernel of D^q is spanned by the integer powers t^n, $0 \leq n < q$ (see formula (4)). Thus, for $q \notin \mathbb{N}$, no real impulse response can be obtained via superposition with a kernel–function, if the branch cut is not $\mathbb{R}_0^-$. Consequently any other branch cut is not physically consistent.

4. Conclusion

The ΨDO–approach is an appropriate tool to analyze and to solve fractional differential equations, due to its mathematical stringency, its easy handling and its applicability. In view to physical consistency, it is translation invariant and includes the past of some system by nature, such that it is predestinated for modelling memory. We have sketched moreover some necessary conditions: Causality and correct modelling of stability require a sufficiently large function space as well as the location of the branch cut in the left half plane. Principal branch definitions of all fractional powers of the symbol and cut $\mathbb{R}_0^-$ are obligatory to obtain real, continuous solutions. We add that all over all the ΨDO approach has proved its efficiency along experiments ([5]).

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