Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 3-4

Blaschke distribution with respect to convolution

Vesna Manova-Eraković

Presented at Internat. Congress "MASSEE' 2003", 4th Symposium "TMSF"

In this work it is proven that the convolution of a distribution T in E' and an upper Blaschke distribution B^+ in D' is again an upper Blaschke distribution in D'. We give also some other properties of the Blaschke product with respect to convolution.

AMS Subj. Classification: 30D50, 46Fxx, 44A35

Key Words: Blaschke product, Blaschke distribution, convolution

1. Introduction

1.1. Denotations which will be used in the paper

Let $\mathcal{U}$ denote the open unit disk in $\mathbb{C}$ i.e. $\mathcal{U} = \{z \in \mathbb{C} \,|\, |z| < 1\}$, $T = \partial\mathcal{U}$ and Π^+ denote the upper half plane i.e. $\Pi^+ = \{z \in \mathbb{C} \,|\, Im\, z > 0\}$.

For a given function f which is analytic on some region Ω we will write $f \in H(\Omega)$.

For a function f, $f : \Omega \to \mathbb{C}^n$, $\Omega \subseteq \mathbb{R}^n$ $x \in \Omega$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N} \cup \{0\}$, $D^\alpha f = D_x^\alpha f(x)$ denotes the differential operator

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

$L^p(\Omega)$ is the space of locally integrable functions on Ω, i.e. $f(x) \in L^p_{loc}(\Omega)$ if $f(x) \in L^p(\Omega')$, for every bounded subregion Ω' of Ω. We denote by $H^\infty(\mathcal{U})$ the space of bounded analytic functions in U. If $f \in H^\infty(\mathcal{U})$, then the radial boundary function

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

is defined almost everywhere on T with respect to the Lebesque measure on T.

1.2. Some notions of distributions

Let Ω be an open subset of $\mathbb{R}^n$. $\mathbb{C}^\infty(\Omega)$ denotes the space of all complex valued infinitely differentiable functions on Ω and $\mathbb{C}_0^\infty(\Omega)$ denotes the subspace of $\mathbb{C}^\infty(\Omega)$ that consists of those functions of $C^\infty(\Omega)$ which have compact support. Support of a function f, denoted by $supp(f)$, is the closure of $\{x \in \Omega | f(x) \neq 0\}$ in Ω.

$D(\Omega)$ denotes the space of $\mathbb{C}_0^\infty(\Omega)$ functions in which convergence is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in C_0^\infty(\Omega)$ converges to $\varphi \in \mathbb{C}_0^\infty(\Omega)$ in $D(\Omega)$ as $\lambda \to \lambda_0$ if and only if there is a compact set $K \subset \Omega$ such that $supp(\varphi_\lambda) \subseteq K$ for each λ, $supp(\varphi) \subseteq K$ and for every n-tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $\{D_t^\alpha \varphi(t)\}$ uniformly on K as $\lambda \to \lambda_0$.

$D'(\Omega)$ is the space of all continuous, linear functionals on $D(\Omega)$, where continuity means that $\varphi_\lambda \to \varphi$ in $D(\Omega)$ as $\lambda \to \lambda_0$, implies $\langle T, \varphi_\lambda \rangle \to \langle T, \varphi \rangle$, as $\lambda \to \lambda_0$, $T \in D'(\Omega)$.

Note. $\langle T, \varphi \rangle$ denotes the value of the functional T, when it acts on the function φ.

A distribution $T \in D'(\Omega)$ equals zero on an open set $\Omega' \subseteq \Omega$ if $< T, \varphi >= 0$ for every $\varphi \in D(\Omega)$ with support in Ω'.

The support of $T \in D'(\Omega)$, denoted by $supp(T)$, is the complement in Ω of the largest open subset of Ω where T equals zero.

For $D(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$, we write D and D', respectively.

$E = E(\mathbb{R}^n)$ denotes the space of $C_0^\infty(\mathbb{R}^n)$ functions in which convergence is defined in the following way: a sequence $\{\varphi_\lambda\}$ of function $\varphi_\lambda \in E$ converges to $\varphi \in E$ in E as $\lambda \to \lambda_0$ if and only if for every $n-$tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $D_t^\alpha \varphi(t)$ uniformly on every compact subset K of $\mathbb{R}^n$ as $\lambda \to \lambda_0$.

E' is the space of all continuous linear functionals on E and the elements of E' are distributions with compact support.

Next, we will mention the notion of the direct product of distributions, the convolution of distribution and some known results which will be used.

Let $T \in D'(O_1)$ and $\mathcal{U} \in D'(O_2)$ and $O_1 \subset \mathbb{R}^n$, $O_2 \subset \mathbb{R}^m$ be open subsets. The unique distribution, denoted by $T \otimes \mathcal{U}$, in $D'(O_1 \times O_2)$, defined by

$$\Big\langle T \otimes \mathcal{U}, \varphi(x,y) \Big\rangle = \Big\langle T_x, < \mathcal{U}_y, \varphi(x,y) > \Big\rangle$$

$$= \Big\langle \mathcal{U}_y, < T_x, \varphi(x,y) > \Big\rangle, \quad \varphi \in D(O_1 \times O_2)$$

is called direct (tensor) product of $T \in D'(O_1)$ and $\mathcal{U} \in D'(O_2)$.

With the use of the direct product for $m = n$, it is defined one of the most important operations in the space of distributions, namely, that of convolution.

At the beginning, we consider the spaces of distributions $D'(\mathbb{R}^n)$, and later we give definition of convolution, when working in $D'(\Omega)$.

As we know, if f and g are two locally integrable functions on $\mathbb{R}^n$ such that one of them has compact support, their convolution is a function $h = f * q$ defined with

$$h(x) = (f * g)(x) = \int\limits_{\mathbb{R}^n} f(x-y)g(y)dy = \int\limits_{\mathbb{R}^n} f(y)g(x-y)dy.$$

Let T_f and T_g be the regular distributions generated by f and g respectively, and $\varphi \in D(\mathbb{R}^n)$. By taking the last equation into account and using Fubini's theorem, we have:

$$\left\langle T_f * T_g, \varphi \right\rangle = \left\langle T_f \otimes T_g, \varphi(x+y) \right\rangle, \quad \varphi \in D(\mathbb{R}^n) \tag{1.2.1}$$

For arbitrary T and $\mathcal{U}$ in $D' = D'(\mathbb{R}^n)$, the equation (1.2.1) can be written formally as

$$\left\langle T * \mathcal{U}, \varphi \right\rangle = \left\langle T \otimes \mathcal{U}, \varphi(x+y) \right\rangle, \quad \varphi \in D. \tag{1.2.2}$$

When at least one of T and $\mathcal{U}$ has compact support, the second term in (1.2.2) is well defined. In general the right side of (1.2.2) may not exist for arbitrary T and $\mathcal{U}$ in $D'(\mathbb{R}^n)$ since $\varphi(x+y)$ as a function of (x, y) does not have compact support in $\mathbb{R}^n \times \mathbb{R}^n$.

So, the convolution $T*\mathcal{U}$ of two distributions $T, \mathcal{U} \in D'(\mathbb{R}^n)$, is defined by (1.2.2), provided that the right side of (1.2.2) is well defined and then $T*\mathcal{U} \in D'$. In that case the convolution can be computed as

$$\left\langle T * \mathcal{U}, \varphi \right\rangle = \left\langle T_x, < \mathcal{U}_y, \varphi(x+y) > \right\rangle = \left\langle \mathcal{U}_y, < T_x, \varphi(x+y) > \right\rangle, \ \varphi \in D. \tag{1.2.3}$$

If one of the distributions T and $\mathcal{U}$ is an element of $E' = E'(\mathbb{R}^n)$, then the convolution $T * \mathcal{U}$ exists and can be computed by (1.2.3).

If instead of U, we consider a function g of D, then $T * g$ is infinitely differentiable function on $\mathbb{R}^n$, i.e. if $T \in D'$, $g \in D$, then $T * g$ is defined with

$$(T * g)(t) = < T_x, g(t-x) > \text{ and } (T * g)(t) \in C^\infty(\mathbb{R}^n).$$

Now, let O be an arbitrary open subset of $\mathbb{R}^n$, $\varepsilon > 0$ be a small positive real number,

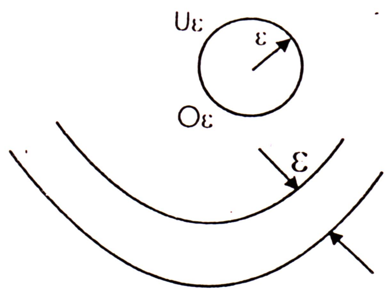
$$\mathcal{U}_\varepsilon = \{x \in \mathbb{R}^n \,|\, |x| < \varepsilon\} \text{ and } O_\varepsilon = \{x \in O \,|\, d(x, \partial O) > \varepsilon\}, (d(x, \partial O) = \inf_{y \in \partial O} |x - y|).$$

Let T and $\mathcal{U}$ belong to $D'(O)$, such that one of them, let us say $\mathcal{U}$, has a compact support in O, such that $supp(\mathcal{U}) \subset \mathcal{U}_\varepsilon$ and $O_\varepsilon \neq \emptyset$ (as on **Picture 1** below), then in [8], it is shown that with

$$\Big\langle T * \mathcal{U}, \varphi \Big\rangle = \Big\langle T \otimes \mathcal{U}, \eta(y)\varphi(x+y) \Big\rangle, \quad \varphi \in D(O_\varepsilon),$$

where $\eta(y) \in D(O_\varepsilon)$, $\eta(y)$ equals 1 on a neighborhood of $supp(\mathcal{U})$, is defined convolution $T * \mathcal{U}$ in $D'(O_\varepsilon)$.

Now, if instead of $\mathcal{U} \in D'(O)$, we consider $\alpha \in D(\mathcal{U}_\varepsilon)$, we get representation for $T * \alpha$: $(T * \alpha)(x) = \Big\langle T_y, \alpha(x-y) \Big\rangle$, $\quad x \in O_\varepsilon$ and $T * \alpha \in \mathbb{C}^\infty(O_\varepsilon)$.



Picture 1

1.3. Blaschke products, Blaschke distributions and some of their properties

Let $\{z_n\}$ be a sequence of points in Π^+ such that

$$\sum_{n=1}^{\infty} \frac{y_n}{1+|z_n|^2} < \infty, \quad z_n = x_n + iy_n \in \Pi^+. \tag{1.3.1}$$

Let m be the number of z_n equal to 0. Then the infinite product

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_{n=1}^{\infty} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\overline{z_n}} \tag{1.3.2}$$

converges on Π^+. The function $B(z)$ of the form (1.3.2) is called Blaschke product.

$B(z)$ is in $H^\infty(\Pi^+)$, and the zeros of $B(z)$ are precisely the points z_n, each zero having multiplicity equal the number of times it occurs in the sequence $\{z_n\}$.

Note: If the number of zeros z_n in (1.3.2) is finite, then we call $B(z)$ finite Blaschke product.

Let $B(z)$ be a Blaschke product, $z = x + iy \in \Pi^+$, with zeros $\{z_n\}$ that belong to the upper half plane Π^+.

By $< B^+, \varphi >$ we denote:

(1.3.3) $$< B^+, \varphi >= \lim_{y\to 0^+} \int_{-\infty}^{+\infty} B(z)\varphi(x)dx, \quad z = x + iy \in \Pi^+, \ \varphi \in D(\mathbb{R}).$$

In [5], it is proven that B^+ is a distribution on D, called upper Blaschke distribution on D.

Similarly, for $B(z)$ being a Blaschke product with zeros z_n, $n \in \mathbb{N}$ that belong to the lower half plane Π^-, by $< B^-, \varphi >$, for $\varphi \in D$ we denote

(1.3.4) $$< B^-, \varphi >= \lim_{y\to 0^+} \int_{-\infty}^{+\infty} B(z)\varphi(x)dx, \quad z = x - iy \in \Pi^-, \quad \varphi \in D(\mathbb{R}).$$

B^- is distribution on D, called lower Blaschke distribution on D.

Note. In the above definitions, when $B(z)$ is finite Blaschke product, we call the associate Blaschke distribution finite Blaschke distribution.

The Blaschke distribution is a new notion in the theory of distributions and has useful applications in the problems of approximation. The introduced Blaschke distributions in [5] were used for representing some distributions in D' as a limit of sequence of Blaschke distributions. In [6], it is given another application of the Blaschke distribution concerning approximation, i.e. some regular distribution in $S'(\mathbb{R})$ are approximated by finite, convex, linear combinations of Blaschke distributions.

2. Main results

Theorem 1. *Let $T \in E'$ and B^+ be an upper Blaschke distribution in D'. Then $B^+ * T$ is upper Blaschke distribution on D.*

Proof. Let $T \in E'$ and B^+ be an upper Blaschke distribution in D'. Then there exists Blaschke product $B(x + iy_1)$ with zeros $z_n \in \Pi^+$, $n \in \mathbb{N}$, such that

(2.0.1) $$< B^+, \varphi >= \lim_{y_1\to 0^+} \int_{-\infty}^{+\infty} B(x + iy_1)\varphi(x)dx, \quad \varphi \in D.$$

Since, $T \in E'$, the convolution $B^+ * T$ exists and according to the definition of convolution, for $\varphi \in D$, we have that

(2.0.2) $$< B^+ * T, \varphi >=< B^+, < T_y, \varphi(x + y) >>$$

Let us consider the function

$$\Psi(x) =< T_y, \varphi(x + y) > .$$

First of all, we will show that $\Psi(x)$ is infinitely differentiable function.

The expression $\dfrac{\varphi(x+y+h)-\varphi(x+y)}{h}$ converges to $\varphi'(x+y)$ in D, as $h \to 0$. From here, we get

$$\frac{d\Psi}{dx} = \frac{d}{dx} < T_y, \varphi(x+y) >$$

$$= \lim_{h\to 0} \frac{1}{h}[< T_y, \varphi(x+y+h) > - < T_y, \varphi(x+y) >] \overset{T \text{ is linear}}{=}$$

$$= \lim_{h\to 0} \Big\langle T_y, \frac{\varphi(x+y+h)-\varphi(x+y)}{h} \Big\rangle \overset{T \text{ is continuous}}{=}$$

$$= \Big\langle T_y, \lim_{h\to 0} \frac{\varphi(x+y+h)-\varphi(x+y)}{h} \Big\rangle = \Big\langle T_y, \varphi'(x+y) \Big\rangle = -\Big\langle T'_y, \varphi(x+y) \Big\rangle.$$

Similarly, we get that

$$\frac{d^n\Psi}{dx^n} = (-1)^n \Big\langle T_y^{(n)}, \varphi(x+y) \Big\rangle, \quad \text{for every } n \in \mathbb{N}, \quad \text{i.e.,}$$

$$\Psi(x) \in \mathbb{C}^\infty(\mathbb{R}). \tag{2.0.3}$$

In the following, we will show that $\Psi(x) \in D$.

Since $T \in E'$, it follows that T has a compact support. Let K_T be the support of T, K_φ be the support of φ and $\alpha(t)$ be a function of $\mathbb{C}^\infty(\mathbb{R})$ equals 1 on K_T and equals zero outside a neighborhood $O(K_T)$ of K_T. Then

$$\Big\langle T_y, \varphi(x+y) \Big\rangle = \Big\langle T_y, \alpha(y)\varphi(x+y) \Big\rangle.$$

The support of $\alpha(y)$ is contained in $O(K_T)$. The support of $\varphi(x+y)$ is the set $\{y \mid x+y \in K_\varphi\}$. (We get this set via translation of K_φ by $-x$). The intersection $O(K_T) \cap \{y \mid x+y \in K_\varphi\}$ is empty set, for x which do not belong to some compact set. That means that, for fixed x, $\alpha(y)\varphi(x+y) = 0$, for all $y \in \mathbb{R}$. The last is true, because of the following: if $y \in \mathbb{R} \setminus$ *supp* α, then $\alpha(y) = 0$; if $y \in supp(\alpha)$, then $y \in O(K_T)$, and since $O(K_T) \cap \{y \mid x+y \in K_\varphi\} = \emptyset$, we have that $y \notin \{y \mid x+y \in K_\varphi\}$ i.e. $\varphi(x+y) = 0$.

In any case, we get $\alpha(y)\varphi(x+y) = 0$, for all $y \in \mathbb{R}$ (for the fixed x). That means that:

$$\Big\langle T_y, \varphi(x+y) \Big\rangle = \Big\langle T_y, \alpha(y)\varphi(x+y) \Big\rangle = 0.$$

So, we got that the function $\Big\langle T_y, \varphi(x+y) \Big\rangle$ equals zero for all x, which do not belong to some compact set, i.e.

$$\Psi(x) = \Big\langle T_y, \varphi(x+y) \Big\rangle \quad \text{has a compact support}. \tag{2.0.4}$$

From (2.0.3) and (2.0.4), it follows that $\Psi(x) \in D$. Now (2.0.2), gets the form

$$\left\langle B^+ * T, \varphi \right\rangle = \left\langle B^+, \Psi(x) \right\rangle, \quad \Psi(x) \in D \text{ , i.e.}$$

$$\left\langle B^+ * T, \varphi \right\rangle = \lim_{y_1 \to 0+} \int_{-\infty}^{+\infty} B(x + iy_1)\Psi(x)dx, \quad \varphi \in D,$$

is upper Blaschke distribution in D'. ■

Theorem 2. *Let $\{B_n(z)\}$ be a sequence of Blaschke products with zeros in the upper half plane, such that converges to function $f(z) \in H^\infty(\Pi^+)$, uniformly on a compact subsets of Π^+, as $n \to \infty$. Let $T \in E'(\Pi^+)$ with support in Π^+, such that supp$(T) \subset \mathcal{U}(z_0, \varepsilon)$, where $\varepsilon > 0$ is a small, positive real number, $\Pi_\varepsilon = \{z \in \Pi^+ \mid Im\ z > \varepsilon\}$, $z_0 \in \Pi_\varepsilon$ and $\mathcal{U}(z_0, \varepsilon) = \{z \in \Pi^+ \mid |z - z_0| < \varepsilon\} \subset \Pi_\varepsilon$. Then the sequence $\{T * B_n\}$ converges to $T * f$, uniformly on compact subsets of $\Pi_{z_0,\varepsilon} = \{z \in \Pi^+ \mid Im\ z > Im\ z_0 + \varepsilon\}$, as $n \to \infty$.*

P r o o f. Let the conditions of the theorem be satisfied. According to the part 1.2, the convolutions $T * B_n$ and $T * f$ exist, they are infinitely differentiable functions on Π_ε and they are given by:

$$(T * B_n)(t) = \left\langle T_z, B_n(t - z) \right\rangle, \quad t \in \Pi_\varepsilon, \quad n \in \mathbb{N} \text{ and}$$

$$(T * f)(t) = \left\langle T_z, f(t - z) \right\rangle, \quad t \in \Pi_\varepsilon.$$

Let K be a compact subset of $\Pi_{z_0,\varepsilon}$, arbitrary chosen, and $t \in K$, $z \in \Pi_\varepsilon$. Then

$$|(T * B_n)(t) - (T * f)(t)| = |\left\langle T_z, B_n(t - z) \right\rangle - \left\langle T_z, f(t - z) \right\rangle|$$
$$= |\left\langle T_z, B_n(t - z) - f(t - z) \right\rangle|.$$

If $z \in \mathcal{U}_{z_0,\varepsilon}$, according to the assumptions we made in the theorem, $t - z$ belongs to some compact subset K_1 of Π^+. Now, using the fact that $\{B_n(z)\}$ converges to $f(z)$, uniformly on compact subsets of Π^+, as $n \to \infty$, the compactness of $K_1 \subset \Pi^+$, we get that $\{B_n(t - z)\}$ converges uniformly to $f(t - z)$, on K_1, as $n \to \infty$. Finally, using the continuity of the distribution $T \in E'(\Pi^+)$, we get that

$$|\left\langle T_z, B_n(t - z) - f(t - z) \right\rangle| \to 0 \text{ as } n \to \infty, \quad \forall t \in K,$$

$$|\left\langle T * B_n)(t) - (T * f)(t) \right\rangle| \to 0 \text{ as } n \to \infty, \quad \forall t \in K.$$

If $z \in \Pi^+ \setminus U_{z_0,\varepsilon}$, then T equals zero and the conclusion of the theorem is trivial. ■

Corollary. *Let $T \in E'(\Pi^+)$ with support in Π^+, such that $supp(T) \subset \mathcal{U}(z_0, \varepsilon)$, where $\varepsilon > 0$ is a small positive real number, $\Pi_\varepsilon = \{z \in \Pi^+ \mid Im\ z > \varepsilon\}$, $z_0 \in \Pi_\varepsilon$ and $\mathcal{U}(z_0, \varepsilon) = \{z \in \Pi^+ \mid |z - z_0| < \varepsilon\} \subset \Pi_\varepsilon$. Let $f \in H(\Pi^+)$ be such that $|f(z)| < 1$, $z \in \Pi^+$, $|f^*(x)| \leq 1$, $x \in \mathbb{R}$. Then there exists a sequence $\{B_n(z)\}$ of finite Blaschke products with zeros in the upper half plane, which converges to the function $f(z)$, uniformly on compact subsets of Π^+, as $n \to \infty$ and for which the sequence $\{T * B_n\}$ converges to $T * f$, uniformly on compact subsets of $\Pi_{z_0,\varepsilon} = \{z \in \Pi^+ \mid Im\ z > Im\ z_0 + \varepsilon\}$, as $n \to \infty$.*

The proof of the corollary is an immediate consequence of Theorem 2 and another theorem, given in [3] which says that every function $f \in H(\Pi^+)$, $|f(z)| < 1$, $z \in \Pi^+$, $|f^*(x) \leq 1$, $x \in \mathbb{R}$ can be approximated by a sequence of finite Blaschke products uniformly on each compact subset of Π^+.

References

[1] G. Bremerman. *Distributions, Complex Variables and Fourier Transforms*, Mir, Moscow, 1968 (In Russian).

[2] R. D. Carmichael, D. Mitrović. *Distributions and Analytic Functions*, J. Wiley and Sons, Inc., N. York, 1998.

[3] P. L. Duren. *Theory of H^p Spaces*, Acad. Press, N. York, 1970.

[4] J. Garnet. *Bounded Analytic Functions*, Acad. Press, N. York, 1981.

[5] V. Manova - Eraković. Introduction of Blaschke distributions and approximation of distributions in D' by a sequence of finite Blaschke distributions, *Mat. Bilt.* **24[L]**, Skopje, 2000.

[6] V. Manova - Eraković. Approximation of some regular distributions in $S'(\mathbb{R})$ by the finite, convex, linear combinations of Blaschke distributions, Accepted for publication in: *Archivum Mathematicum, Brno.*

[7] L. Schwartz. *Theorie des distributions*, Tome **I**, Tome **II**, Herman and $\mathbb{C}^{ie}$, Paris, 1950-1951.

[8] V. S. Vladimirov. *Generalized Functions in Mathematical Physics*, Nauka, Moscow, 1964.

Institute of Mathematics
Faculty of Natural Sciences and Mathematics
P.O. Box 162, 1000 Skopje, MACEDONIA

e-mail: vesname@iunona.pmf.ukim.edu.mk

Received: 30.09.2003