Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 3-4

Interpolation in the Class M^p, $p > 1$

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Presented at Internat. Congress "MASSEE' 2003", 4th Symposium "TMSF"

In this paper we give sufficient conditions for sequence of complex numbers $(\lambda_k)_{k=1}^{\infty}$, $(c_k)_{k=1}^{\infty}$ so that the interpolation problem has a solution in the class M^p, $p > 1$.

AMS Subj. Classification: 30E05

Key Words: holomorphic function, interpolation, M^p-space

1. Definition of some spaces of holomorphic functions and some known results

Let $D = \{z \in C \mid |z| < 1\}$ be the unit disk. We denote by M^p the space of holomorphic function in D such that

$$\int_0^{2\pi} (\log^+ Mf(\theta))^p d\theta < \infty,$$

where

$$Mf(\theta) = \sup_{0 \le r < 1} |f(re^{i\theta})|$$

and

$$\log^+ a = \begin{cases} 0, & 0 < a \le 1 \\ \log a, & a \ge 1 \end{cases} \quad \text{(see [1])}.$$

By N^p we denote the set of holomorphic functions on the unit disk D such that

$$\sup_{0 \le r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta < \infty.$$

By H^p we denote the set of all holomorphic function on the unit disk such that

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

The following inclusions are true:

$$\bigcup_{p>0} H^p \subset \bigcap_{p>1} N^p \ , \ \bigcup_{p>1} N^p \subset M \subset N.$$

For $p > 1$ it holds $N^p = M^p$. Also, it holds $N \neq M$ (see [3]).

We denote by $b(z)$ the Blaschke product

$$b(z) = \prod_{j=1}^{\infty} \frac{\lambda_j - z}{1 - \overline{\lambda_j} z} \frac{|\lambda_j|}{\lambda_j}, \quad \sum_{j=1}^{\infty} (1 - |\lambda_j|) < \infty, \quad z \in D$$

and

$$b_k(z) = \prod_{j \neq k} \frac{\lambda_j - z}{1 - \overline{\lambda_j} z} \frac{|\lambda_j|}{\lambda_j}.$$

For the Blaschke product it holds $|b(z)| \leq 1$, for $z \in D$ (see [4]).

Let X be a class of holomorphic functions defined on the unit disk and $(\lambda_k)_{k=1}^{\infty}$ be a sequence in D. For the sequence $(c_k)_{k=1}^{\infty}$ of complex numbers, the interpolation problem is the problem of finding the function $f \in X$ such that

$$f(\lambda_k) = c_k \quad \text{for} \quad k \in \mathbb{N}. \tag{1.1}$$

Let Y be a family of sequences of complex numbers. The sequence $(\lambda_k)_{k=1}^{\infty}$ is the interpolation sequence of the pair (X, Y), if for every sequence $(c_k)_{k=1}^{\infty} \subset Y$ exists function $f \in X$ such that (1.1) holds.

The sequence is uniformly separated, if there is positive number δ such that

$$\inf_{k \in \mathbf{N}} |b_k(\lambda_k)| \geq \delta.$$

By l^{∞} we denote the set of all bounded sequences of complex numbers.

Carleson (1958) has proved the following result:

The sequence $(\lambda_k)_{k=1}^{\infty}$, $\lambda_k \in D$ for which $\sum_1^{\infty}(1 - |\lambda_k|) < \infty$ is the interpolation sequence for the pair (H^{∞}, l^{∞}), if and only if it is uniformly separated.

Let (λ_n) be a sequence of complex numbers such that $\lambda_k \in D, \ \ \lambda_n \neq \lambda_m$, for $n \neq m$ and $\sum_1^{\infty}(1 - |\lambda_k|) < \infty$.

For all $0 < p < \infty$ with $\overline{l}^p$ we denote the set of all sequences of complex numbers (c_k) such that $\sum_1^{\infty}(1 - |\lambda_k|^2)|c_k|^p < \infty$.

Let $0 < p < \infty$. The sequence $(\lambda_k)_{k=1}^{\infty}$, $\lambda_k \in D$ for which $\sum_1^{\infty}(1 - |\lambda_k|) < \infty$, is the interpolation sequence for the pair $(H^p, \bar{l}^p)$, if and only if it is uniformly separated.

The above result for $1 \leq p < \infty$ is obtained by H.S. Shapiro and A.L. Shields (1961) and for $0 < p < 1$ is obtained by V. Kabaila (1963).

We give a condition for these sequences $(\lambda_k)_{k=1}^{\infty}$ and $(c_k)_{k=1}^{\infty}$ so that a function $f \in M^p$ with the property $f(\lambda_k) = c_k$, $k = 1, 2, \ldots$. exists.

We will use the next theorems.

Theorem A. *The sum of a uniformly convergent series of holomorphic functions is a holomorphic function in every inner point on the set, where the series converges uniformly, see* [5].

Theorem B. *(Principle for compactness for holomorphic functions) Let (f_n) be a sequence of functions which are holomorphic on region G and for which it holds: for every closed set $B \subset G$ exists a constant $M(B)$ such that for every function of the sequence (f_n) it is true*

$$|f_n(z)| \leq M(B) \quad \textit{for every} \quad z \in B.$$

Then the sequence (f_n) has subsequence which converges uniformly on every closed subset from G, see [5]

Holder's inequality. *If a_k and b_k are complex numbers, $k = 1, \ldots, n$ and if $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then* $\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}}$.

2. Main results

Theorem 1. *Let $p > 1$ and let $(\lambda_k)_{k=1}^{\infty}$ be a sequence of complex numbers such that $\lambda_k \in D$ and $\sum_{k=1}^{\infty}(1 - |\lambda_k|)^{\frac{1}{p}} < \infty$. Let $(c_k)_{k=1}^{\infty}$ be a sequence of complex numbers such that*

$$\sum_{k=1}^{\infty} \frac{(1 - |\lambda_k|)^{\varepsilon}}{|b_k(\lambda_k)|} |\log c_k| < \infty \quad \textit{for some } \varepsilon \textit{ which satisfies } 0 < \varepsilon < 1.$$

Then there exists $f \in M^p$ such that $f(\lambda_j) = c_j$ for $j \in \mathbb{N}$ and $f \notin H^s$, for every $s > 0$.

P r o o f. Because of $0 \le |\lambda_k| < 1$ and $p > 1$, it follows that $\sum_{k=1}^{\infty}(1-|\lambda_k|) < \infty$. Let $f(z) = \exp g(z)$, where $g(z) = g_1(z) + g_2(z) + g_3(z)$ and

$$g_1(z) = \sum_{k=1}^{\infty} \frac{(1-|\lambda_k|)^{\alpha+1}}{(1-ze^{-i\theta_k})^{\alpha}} \quad g_2(z) = -\sum_{k=1}^{\infty}(1-|\lambda_k|),$$

$$g_3(z) = \sum_{k=1}^{\infty} \frac{b_k(z)}{b_k(\lambda_k)} \cdot \frac{(1-|\lambda_k|)^{\varepsilon}}{(1-ze^{-i\theta_k})^{\varepsilon}} \cdot \log c_k,$$

where $\theta_k = \arg \lambda_k$ and $\alpha = \dfrac{1}{p}$.

It is obvious that $f(\lambda_j) = c_j, \;\; j \in \mathbb{N}$.

Since f is a holomorphic function, it is sufficient to prove that $g(z)$ is holomorphic function. This is a corollary of the above expansions via Theorem A. Further, we get the following estimates:

$$|g_1(z)| \le \sum_{k=1}^{\infty} \left| \frac{(1-|\lambda_k|)^{\alpha+1}}{(1-ze^{-i\theta_k})^{\alpha}} \right| \le \sum_{k=1}^{\infty} \frac{(1-|\lambda_k|)^{\alpha+1}}{(1-|z|)^{\alpha}} \le \frac{A}{(1-|z|)^{\alpha}} \quad \text{for some } A > 0,$$

$$|g_3(z)| \le \sum_{k=1}^{\infty} \frac{|b_k(z)|}{|b_k(\lambda_k)|} \cdot \frac{(1-|\lambda_k|)^{\varepsilon}}{|1-ze^{-i\theta_k}|^{\varepsilon}} \cdot |\log c_k|$$

$$\le \sum_{k=1}^{\infty} \frac{1}{|b_k(\lambda_k)|} \cdot \frac{(1-|\lambda_k|)^{\varepsilon}}{|1-ze^{-i\theta_k}|^{\varepsilon}} \cdot |\log c_k| \le \frac{B}{(1-|z|)^{\varepsilon}} \text{ for some } B > 0.$$

Let $s > 0$. Let $f_1(z) = \exp g_1(z),\ f_2(z) = \exp g_3(z),\ A_k = (1-|\lambda_k|)^{\alpha+1}$. We will estimate $|f_1(z)|$:

$$|f_1(z)| = \exp\Big(Re \sum_{k=1}^{\infty} \frac{(1-|\lambda_k|)^{\alpha+1}}{(1-ze^{-i\theta_k})^{\alpha}} \Big)$$

$$= \exp(\sum_{k=1}^{\infty} A_k Re\Big(\frac{1}{(1-ze^{-i\theta_k})^{\alpha}}\Big) = \exp(\sum_{k=1}^{\infty} A_k Re \frac{(1-re^{-i(\theta-\theta_k)})^{\alpha}}{(1-2r\cos(\theta-\theta_k)+r^2)^{\alpha}}\Big)$$

$$= \exp(\sum_{k=1}^{\infty} A_k Re \frac{(1-\alpha re^{-i(\theta-\theta_k)} + \binom{\alpha}{2} r^2 e^{-2i(\theta-\theta_k)} - \ldots)}{(1-2r\cos(\theta-\theta_k)+r^2)^{\alpha}}$$

$$= \exp\Big(\sum_{k=1}^{\infty} A_k \frac{1-\alpha r\cos(\theta-\theta_k) + \binom{\alpha}{2} r^2 \cos 2(\theta-\theta_k) - \ldots}{(1-2r\cos(\theta-\theta_k)+r^2)^{\alpha}}\Big)$$

$$\geq \exp\Big(\sum_{k=1}^{\infty} A_k \frac{1 - \alpha r - \binom{\alpha}{2} r^2 - \dots}{(1 - 2r\cos(\theta - \theta_k) + r^2)^\alpha}\Big)$$

$$= \exp\Big(\sum_{k=1}^{\infty} A_k \frac{2 - (1 + \alpha r + \binom{\alpha}{2} r^2 + \dots}{(1 - 2r\cos(\theta - \theta_k) + r^2)^\alpha}\Big)$$

$$= \exp\Big(\sum_{k=1}^{\infty} A_k \frac{2 - (1 + r)^\alpha}{(1 - 2r\cos(\theta - \theta_k) + r^2)^\alpha}\Big)$$

$$\geq \exp\left((2 - (1 + r)^\alpha\right) \frac{A_1}{(1 - 2r\cos(\theta - \theta_k) + r^2)^\alpha}.$$

We use the inequality $e^x \geq \dfrac{x^\beta}{\beta!}$, $\beta > 0$ and $x > 0$. Now,

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f_1(re^{i\theta})|^s d\theta$$

$$\geq \sup_{0 \leq r < 1} \int_0^{2\pi} \exp(s(2 - (1 + r)^\alpha) \frac{A_1}{(1 - 2r\cos(\theta - \theta_k) + r^2)^\alpha} d\theta$$

$$\geq \sup_{0 \leq r < 1} \int_0^{2\pi} \frac{1}{(p!)} \frac{s^p (2 - (1 + r)^\alpha)^p A_1^p}{1 - 2r\cos(\theta - \theta_k) + r^2)^{\alpha p}} d\theta.$$

Since $\alpha = \dfrac{1}{p}$, then

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f_1(re^{i\theta})|^s d\theta \geq \frac{s^p}{p!} A_1^p \sup_{0 \leq r < 1} \int_0^{2\pi} \frac{(2 - (1 + r)^{\frac{1}{p}})^p}{1 - 2r\cos\theta + r^2} d\theta$$

$$= \frac{s^p}{p!} A_1^p 2\pi \sup_{0 \leq r < 1} \frac{2 - (1 + r)^{\frac{1}{p}})^p}{1 - r^2} = \infty.$$

Since $f_2(z) = \exp g_3(z)$ has no zero, it exists $\delta_0 > 0$ such that:

$$|f_2(z)| \geq \delta_0 > 0, \ \ \forall z \in \overline{D} = \{z \ : \ |z| \leq 1\}.$$

So we have

$$\int_0^{2\pi} |f(re^{i\theta})|^s d\theta = l^s \int_0^{2\pi} |f_1(re^{i\theta})|^s |f_2(re^{i\theta})|^s d\theta \geq \delta_0 l^s \int_0^{2\pi} |f_1(re^{i\theta})|^s d\theta = \infty,$$

where $l = \exp g_2(z)$.

Now we show that $f \in M^p$. We use the inequalities $\log^+ ab \leq log^+ a + log^+ b$ and $(a+b)^p \leq 2^p(a^p + b^p)$ for $a > 0$ and $b > 0$. For $p > 1$ it holds $M^p = N^p$, in the sense of a set.

We show that $f_1 \in M^p$. We have shown above (in the estimation of $|f_1(z)|$) that $Re \sum_{k=1}^{\infty} \frac{A_k}{(1 - ze^{i\theta_k})^\alpha} > 0$ for $|z| < 1$.

Now, the following holds:

$$\log^+ |f_1(z)| = \log^+ \exp \Big(Re \sum_{k=1}^{\infty} \frac{A_k}{(1 - ze^{-i\theta_k})^\alpha} \Big)$$

$$= Re \sum_{k=1}^{\infty} \frac{A_k}{(1 - ze^{-i\theta_k})^\alpha} \leq \sum_{k=1}^{\infty} \frac{A_k}{|1 - ze^{-i\theta_k}|^\alpha} .$$

Let $1 < q' < p$ and $\frac{1}{p'} + \frac{1}{q'} = 1$. Then from Holder's inequality we have

$$(\log^+ |f_1(z)|^p \leq \Big(\sum_{k=1}^{\infty} A_k^{p'} \Big)^{\frac{p}{p'}} \cdot \Big(\sum_{k=1}^{\infty} \frac{1}{|1 - ze^{-i\theta_k}|^{\alpha q'}} \Big)^{\frac{p}{q'}} .$$

It is true that: $\sum_{k=1}^{\infty} A_k^{p'} < +\infty$ and $\alpha q' < 1$ $(\alpha = \frac{1}{p})$.

Then, if $z = re^{i\theta}$ $(0 \leq r < 1)$, we get

$$\int_0^{2\pi} (\log^+ |f_1(re^{i\theta})|)^p \leq \sum_{k=1}^{\infty} A_k^{p'} \int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^{\alpha q'}} < C,$$

$$\text{i.e.} \qquad f_1 \in M^p.$$

Since $\sum_{k=1}^{\infty} (1 - |\lambda_k|)^\varepsilon \frac{|\log c_k|}{|b_k(\lambda_k)|} < \infty$, in a similar way, for f_1 we can show $f_2 \in M^p$.

It holds $\exp g_2(z) \in M^p$, so $f \in M^p$. ∎

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Received: 30.09.2003