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Z-Function and Solutions of Fractional Differential Equations

Zaid Odibat [1] *and Ahmed Alawneh* [2]

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Differential equations of fractional order appear more and more frequently in different research areas including physical and engineering applications. An effective technique for solving such equations is needed.

In this work, we introduce a new function called the Z-function, some properties of which are discussed. Some known functions will be given as special cases of Z-function. This work shows that the proposed function, Z-function, is necessary and appropriate for its connection with fractional calculus and solutions of fractional differential equations.

Solution of the initial-value problem for the fractional linear differential equation with constant coefficients based on Riemann-Liouville definition of fractional derivatives of arbitrary order will be given. The solutions obtained are calculated in terms of the Z-function.

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1. Introduction, definitions and preliminaries

We introduce the following definitions and properties of the fractional integrals and Riemann-Liouville fractional derivatives:

Fractional Integral. According to the Riemman-Liouville approach to fractional calculus, the notation of fractional integral of order $\alpha > 0$ is a natural consequence of Cauchy integral formula:

$$f_n(t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau)d\tau, \qquad t > 0, \qquad n \in N. \tag{1.1}$$

In a natural way, if replacing the positive integer n by the positive real number α, one defines the Riemann-Liouville fractional integral of order $\alpha > 0$ as:

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad t > 0. \tag{1.2}$$

For example, if $f(t) = t^\gamma$ with $\gamma > -1$, $\alpha > 0$, then we have

$$D^{-\alpha} t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \qquad t > 0. \tag{1.3}$$

We note the semigroup property

$$D^{-\alpha} D^{-\beta} = D^{-(\alpha+\beta)}, \qquad \alpha, \beta \geq 0, \tag{1.4}$$

which implies the commutative property $D^{-\alpha} D^{-\beta} = D^{-\beta} D^{-\alpha}$.

Riemann-Liouville fractional derivative. Let m be the smallest integer that exceeds α, then the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as:

$$D^\alpha f(t) = D^m \left[D^{-(m-\alpha)} f(t) \right], \tag{1.5}$$

namely

$$D^\alpha f(t) = \left\{ \begin{array}{ll} \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \right], & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{array} \right\}. \tag{1.6}$$

We can easily recognize that, for $\alpha \geq 0$

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \quad \gamma > -1, \quad t > 0. \tag{1.7}$$

Details and properties of this definition are given in [9].

For the standard Riemann-Liouville fractional derivative the Laplace transform, if exists, requires the knowledge of the initial values of the fractional integral $D^{-(m-\alpha)}$ and its integer derivatives of order $k = 0, 1, ..., m-1$. In fact, we have the following rule for the Laplace transform,

$$\int_0^\infty e^{-st} D^\alpha f(t) = s^\alpha \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-k-1} D^{k-m+\alpha} f(0), \quad m-1 < \alpha \leq m. \tag{1.8}$$

2. The Z-function

In this section, we define and discuss some properties of the Z-function, we give a number of useful relationships and the Laplace transform of the Z-function. The proposed function, as we shall show in this study, is relevant for its connection with solutions of linear fractional differential equations.

Definition 2.1. The Z-function

$$\mathbf{Z}_p^\alpha(z) = \mathbf{Z}_p^\alpha\left(z \,\middle|\, \begin{matrix} \alpha_1, \alpha_2, ..., \alpha_k \\ a_1, a_2, ..., a_k \end{matrix}\right), \tag{2.1}$$

where α and α_i are positive real numbers ($\alpha > \alpha_1 > \alpha_2 > ... > \alpha_k$), p and a_i are arbitrary real numbers, is characterized by the series representation

$$\mathbf{Z}_p^\alpha\left(z \,\middle|\, \begin{matrix} \alpha_1 \\ a_1 \end{matrix}\right) = \sum_{n=0}^{\infty}(-1)^n a_1^n \frac{z^{(n+1)\alpha - n\alpha_1 - p}}{\Gamma((n+1)\alpha - n\alpha_1 - p + 1)}, \tag{2.2}$$

$$\mathbf{Z}_p^\alpha\left(z \,\middle|\, \begin{matrix} \alpha_1, \alpha_2, ..., \alpha_k \\ a_1, a_2, ..., a_k \end{matrix}\right) = \sum_{n=0}^{\infty}(-1)^n \sum \binom{n}{n_1, n_2, ..., n_k} a_1^{n_1} a_2^{n_2} ... a_k^{n_k}$$

$$\times \frac{z^{(n+1)\alpha - n_1\alpha_1 - \cdots - n_k\alpha_k - p}}{\Gamma((n+1)\alpha - n_1\alpha_1 - \cdots - n_k\alpha_k - p + 1)}, \tag{2.3}$$

where the second summation extends over all possible combinations of nonnegative integers $n_1, n_2, ..., n_k$ such that $n_1 + n_2 + \cdots + n_k = n$.

For $\alpha > \alpha_1 > \cdots > \alpha_k$, $a_k \neq 0$ and $z \in \mathbb{C}$, the Z-function $\mathbf{Z}_p^\alpha$ is an analytic function, since the convergence radius of series (2.3) is ∞.

Special Cases. It follows from the definition (2.2) that

$$\mathbf{Z}_1^1\left(z \,\middle|\, \begin{matrix} 0 \\ a \end{matrix}\right) = e^{-az}, \tag{2.4}$$

$$\mathbf{Z}_1^2\left(z \,\middle|\, \begin{matrix} 0 \\ a \end{matrix}\right) = \frac{1}{\sqrt{a}} \sin\left(\sqrt{a}z\right), \tag{2.5}$$

$$\mathbf{Z}_2^2\left(z \,\middle|\, \begin{matrix} 0 \\ a \end{matrix}\right) = \cos\left(\sqrt{a}z\right), \tag{2.6}$$

$$\mathbf{Z}_\alpha^\alpha\left(z \,\middle|\, \begin{matrix} 0 \\ a \end{matrix}\right) = E_\alpha\left(-az^\alpha\right), \tag{2.7}$$

where $E_\alpha(z)$ is the Mittag-Leffler function in one parameter.

Identities of Z-function. The Z-function satisfies the following identities:

$$\mathbf{Z}_p^{\alpha}\left(z\,\middle|\,\begin{array}{c}\cdots\beta\cdots\gamma\cdots\\ \cdots a\cdots b\cdots\end{array}\right)=\mathbf{Z}_p^{\alpha}\left(z\,\middle|\,\begin{array}{c}\cdots\gamma\cdots\beta\cdots\\ \cdots b\cdots a\cdots\end{array}\right), \tag{2.8}$$

$$\mathbf{Z}_p^{\alpha}\left(z\,\middle|\,\begin{array}{c}\cdots\beta\cdots\\ \cdots 0\cdots\end{array}\right)=\mathbf{Z}_p^{\alpha}\left(z\,\middle|\,\begin{array}{c}\cdots 0\cdots\\ \cdots 0\cdots\end{array}\right), \tag{2.9}$$

$$\mathbf{Z}_p^{\alpha}\left(z\,\middle|\,\begin{array}{c}\cdots\beta\cdots\beta\cdots\\ \cdots a\cdots b\cdots\end{array}\right)=\mathbf{Z}_p^{\alpha}\left(z\,\middle|\,\begin{array}{cc}\cdots\quad\beta & \cdots 0\cdots\\ \cdots a+b & \cdots 0\cdots\end{array}\right), \tag{2.10}$$

$$\mathbf{Z}_p^{\alpha}\left(z\,\middle|\,\begin{array}{c}\alpha_1,\alpha_2,...,\alpha_k\\ a_1,a_2,...,a_k\end{array}\right)=\mathbf{Z}_{p-\alpha_k}^{\alpha-\alpha_k}\left(z\,\middle|\,\begin{array}{cccc}\alpha_1-\alpha_k, & \alpha_2-\alpha_k, & ..., & \alpha_{k-1}-\alpha_k, 0\\ a_1\;, & a_2\;, & ..., & a_{k-1}\;,\; a_k\end{array}\right), \tag{2.11}$$

where the dots represents any parameters. The proof follows immediately from the Z-function definition.

Laplace transform of Z-function. It is known that Laplace transform is a powerful tool for solving initial value problem for both ordinary and fractional differential equations. In the following theorem, we compute the Laplace transform of the Z-function.

Theorem 2.2. *Suppose that $\alpha>\alpha_1>\alpha_2>\cdots>\alpha_k>0$, $\alpha-p>-1$ and $t>0$, then the Laplace transform of the Z-function is given by:*

$$\int_0^{\infty} e^{-st}\mathbf{Z}_p^{\alpha}\left(t\,\middle|\,\begin{array}{c}\alpha_1,\alpha_2,...,\alpha_k\\ a_1,a_2,...,a_k\end{array}\right)dt=\frac{s^{p-1}}{s^{\alpha}+a_1s^{\alpha_1}+a_2s^{\alpha_2}+\cdots+a_ks^{\alpha_k}}, \tag{2.12}$$

$$\left|\frac{a_1s^{\alpha_1}+a_2s^{\alpha_2}+\cdots+a_ks^{\alpha_k}}{s^{\alpha}}\right|<1. \tag{2.13}$$

P r o o f. From the definition (2.3), it follows that

$$\int_0^{\infty} e^{-st}\mathbf{Z}_p^{\alpha}\left(t\,\middle|\,\begin{array}{c}\alpha_1,\alpha_2,...,\alpha_k\\ a_1,a_2,...,a_k\end{array}\right)dt$$

$$=\int_0^{\infty} e^{-st}\sum_{n=0}^{\infty}\sum(-1)^n\binom{n}{n_1,n_2,...,n_k}a_1^{n_1}a_2^{n_2}...a_k^{n_k}$$

$$\times\frac{t^{(n+1)\alpha-n_1\alpha_1-\cdots-n_k\alpha_k-p}}{\Gamma((n+1)\alpha-n_1\alpha_1-\cdots-n_k\alpha_k-p+1)}\,dt, \tag{2.14}$$

where the second summation extends over all possible combinations of nonnegative integers $n_1, n_2, ..., n_k$ such that $n_1 + n_2 + \cdots + n_k = n$. Performing term by term integration and using the well known property:

$$\int_0^\infty e^{-st} t^\gamma dt = \frac{\Gamma(\gamma+1)}{s^{\gamma+1}}, \qquad \gamma > -1, \tag{2.15}$$

we can write the right hand side of (2.14) in the form

$$\sum_{n=0}^{\infty} (-1)^n \sum \binom{n}{n_1, n_2, ..., n_k} a_1^{n_1} a_2^{n_2} ... a_k^{n_k} \frac{s^{p-\alpha-1}}{s^{n\alpha - n_1\alpha_1 - \cdots - n_k\alpha_k}} \tag{2.16}$$

$$= s^{p-\alpha-1} \sum_{n=0}^{\infty} (-1)^n \sum \binom{n}{n_1, n_2, ..., n_k} \left(\frac{a_1}{s^{\alpha-\alpha_1}}\right)^{n_1} \left(\frac{a_2}{s^{\alpha-\alpha_2}}\right)^{n_2} \cdots \left(\frac{a_k}{s^{\alpha-\alpha_k}}\right)^{n_k}, \tag{2.17}$$

since $n_1 + n_2 + \cdots + n_k = n$.

Using the multinomial theorem and (2.13), we can write (2.14) as:

$$\int_0^\infty e^{-st} \mathbf{Z}_p^\alpha(t) dt = s^{p-\alpha-1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a_1}{s^{\alpha-\alpha_1}} + \frac{a_2}{s^{\alpha-\alpha_2}} + \cdots + \frac{a_k}{s^{\alpha-\alpha_k}}\right)^n \tag{2.18}$$

$$= s^{p-\alpha-1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \cdots + a_k s^{\alpha_k}}{s^\alpha}\right)^n \tag{2.19}$$

$$= \frac{s^{p-1}}{s^\alpha + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \cdots + a_k s^{\alpha_k}}. \tag{2.20}$$

■

3. Fractional differential equations

In this section, we give solutions of initial value problems for linear ordinary fractional differential equations with constant coefficients, where the fractional derivatives are based on the Riemann-Liouville definition.

Theorem 3.1. *Suppose that $m-1 < \alpha \le m$, $\alpha > \alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$, $m_i - 1 < \alpha_i \le m_i$ $(i = 1, 2, \cdots, k)$ and $t > 0$, then the initial value problem*

$$\left[D^\alpha + a_1 D^{\alpha_1} + \cdots + a_k D^{\alpha_k} + a_{k+1} D^0\right] y(t) = f(t), \tag{3.1}$$

where D^α is the Riemann-Liouville fractional derivative of order α, has the following solution

$$
\begin{aligned}
y(t) = {} & \int_0^t f(t-\tau)\mathbf{Z}_1^{\alpha}\left(\tau \,\middle|\, \begin{matrix} \alpha_1, \alpha_2, ..., \alpha_k \ , 0 \\ a_1, a_2, ..., a_k, a_{k+1} \end{matrix}\right) d\tau \\
& + \sum_{i=0}^{m-1} D^{i-m+\alpha} y(0) \mathbf{Z}_{m-i}^{\alpha}\left(t \,\middle|\, \begin{matrix} \alpha_1, \alpha_2, ..., \alpha_k \ , 0 \\ a_1, a_2, ..., a_k, a_{k+1} \end{matrix}\right) \\
& + \sum_{i=0}^{m_1-1} a_1 D^{i-m_1+\alpha_1} y(0) \mathbf{Z}_{m_1-i}^{\alpha}\left(t \,\middle|\, \begin{matrix} \alpha_1, \alpha_2, ..., \alpha_k \ , 0 \\ a_1, a_2, ..., a_k, a_{k+1} \end{matrix}\right) \\
& \vdots \\
& + \sum_{i=0}^{m_k-1} a_k D^{i-m_k+\alpha_k} y(0) \mathbf{Z}_{m_k-i}^{\alpha}\left(t \,\middle|\, \begin{matrix} \alpha_1, \alpha_2, ..., \alpha_k \ , 0 \\ a_1, a_2, ..., a_k, a_{k+1} \end{matrix}\right). \qquad (3.2)
\end{aligned}
$$

Proof. Applying the Laplace transform to both sides of (3.1) and using (1.8), we obtain

$$
\begin{aligned}
& s^{\alpha} \tilde{y}(s) - \sum_{i=0}^{m-1} D^{i-m+\alpha} y(0) s^{m-i-1} + a_1 s^{\alpha_1} \tilde{y}(s) - \sum_{i=0}^{m_1-1} a_1 D^{i-m_1+\alpha_1} y(0) s^{m_1-i-1} \\
& + \cdots + a_k s^{\alpha_k} \tilde{y}(s) - \sum_{i=0}^{m_k-1} a_k D^{i-m_k+\alpha_k} y(0) s^{m_k-i-1} + a_{k+1} \tilde{y}(s) = \tilde{f}(s), \qquad (3.3)
\end{aligned}
$$

that is

$$
\begin{aligned}
\tilde{y}(s) = {} & \frac{\tilde{f}(s)}{s^{\alpha} + a_1 s^{\alpha_1} + \cdots + a_k s^{\alpha_k} + a_{k+1}} \\
& + \frac{\sum_{i=0}^{m-1} D^{i-m+\alpha} y(0) s^{m-i-1}}{s^{\alpha} + a_1 s^{\alpha_1} + \cdots + a_k s^{\alpha_k} + a_{k+1}} + \frac{\sum_{i=0}^{m_1-1} a_1 D^{i-m_1+\alpha_1} y(0) s^{m_1-i-1}}{s^{\alpha} + a_1 s^{\alpha_1} + \cdots + a_k s^{\alpha_k} + a_{k+1}} \\
& + \cdots + \frac{\sum_{i=0}^{m_k-1} a_k D^{i-m_k+\alpha_k} y(0) s^{m_k-i-1}}{s^{\alpha} + a_1 s^{\alpha_1} + \cdots + a_k s^{\alpha_k} + a_{k+1}}, \qquad (3.4)
\end{aligned}
$$

where $\tilde{y}(s)$ and $\tilde{f}(s)$ are the Laplace transforms of $y(t)$ and $f(t)$, receptively. Inverting the Laplace transform, using (2.12), we obtain the solution (3.2) of the problem (3.1). ■

Example 3.2. The initial value problem

$$D^{\alpha}y(t) + ay(t) = 0, \qquad t > 0, \tag{3.5}$$

with intial conditions

$$D^{\alpha-i}y(0) = c_i, \qquad i = 1, 2, \cdots, m, \tag{3.6}$$

where $m - 1 < \alpha \leq m$, has the following solution

$$y(t) = \int_0^t f(t-\tau)\mathbf{Z}_1^{\alpha}\left(\tau \,\middle|\, \begin{matrix} 0 \\ a \end{matrix}\right) d\tau + \sum_{p=1}^{m} c_p \mathbf{Z}_p^{\alpha}\left(t \,\middle|\, \begin{matrix} 0 \\ a \end{matrix}\right) \tag{3.7}$$

$$= \int_0^t f(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}\left(-a\tau^{\alpha}\right) d\tau + \sum_{p=1}^{m} c_p t^{\alpha-p} E_{\alpha,\alpha-p+1}\left(-at^{\alpha}\right). \tag{3.8}$$

Example 3.3. Consider the initial value problem

$$\frac{d}{dt}y(t) + aD^{\beta}y(t) + by(t) = f(t), \qquad t > 0, \tag{3.9}$$

where $0 < \beta < 1$, with initial conditions

$$D^{\beta-1}y(0) = c_1\,, \qquad y(0) = c_2. \tag{3.10}$$

Comparing with (3.1), (3.9) admits the solution

$$y(t) = \int_0^t f(t-\tau)\mathbf{Z}_1^{1}\left(\tau \,\middle|\, \begin{matrix} \beta, 0 \\ a, b \end{matrix}\right) d\tau + c_2\mathbf{Z}_1^{1}\left(t \,\middle|\, \begin{matrix} \beta, 0 \\ a, b \end{matrix}\right) + ac_1\mathbf{Z}_{\beta}^{2}\left(t \,\middle|\, \begin{matrix} \beta, 0 \\ a, b \end{matrix}\right). \tag{3.11}$$

4. Summary

In this work, we introduce a new function called the Z-function in terms of a convergent power series which generalizes some known functions. Some properties and the Laplace transform of the Z-function are also presented.

The study showed that the proposed function, Z-function, is appropriate for its connection with fractional calculus and plays an important role in solving initial value problems for fractional differential equations. Solutions of some known fractional differential equations are given in terms of Z-function.

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[1] *Faculty of Applied Science*
Al-Balqa Applied University, Salt, JORDAN
e-mail: odibat@bau.edu.jo

[2] *Dept. of Mathematics*
University of Jordan, Amman, JORDAN
e-mail: alawneh@sci.ju.edu.jo

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