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Extension of a Finite Many Numbers to an Interpolating Sequence and Approximation by Blaschke Products

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We approximate a finite Blaschke product by interpolating one.

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1. Introduction

Denote by U the unit disk in the plane.

A finite Blaschke product in the unit disk is a holomorphic function of the form

$$b_n(z) = \prod_{k=1}^{n} \frac{\lambda_k - z}{1 - \overline{\lambda}_k z} \frac{|\lambda_k|}{\lambda_k}, \quad z \in U,$$

$\lambda \in U$, $k = 1, 2, \ldots, n$. An infinite Blaschke product in the unit disk is a holomorphic function of the form

$$b(z) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{1 - \overline{\lambda_k} z} \cdot \frac{|\lambda_k|}{\lambda_k}, \sum_{k=1}^{\infty} (1 - |\lambda_k|) < \infty, z \in U, \lambda_k \in U, k = 1, 2, \ldots,$$

The sequence $(\lambda_k)_{k=1}^{\infty} \subset U$ is called an H^∞ interpolating sequence, if for every bounded sequence $(c_k)_{k=1}^{\infty}$ there is bounded holomorphic function such that $f(\lambda_k) = c_k$, $k = 1, 2, \ldots$. By the theorem of Carleson the sequence $(\lambda_k)_{k=1}^{\infty} \subset U$ is an interpolating sequence if there exist $\delta > 0$ such that

$$\inf \prod_{j, k \neq j} \left| \frac{\lambda_j - \lambda_k}{1 - \overline{\lambda}_j \lambda_k} \right| \geq \delta.$$

The Blaschke product is called an interpolating Blaschke product, if its zeros are different and form an interpolating sequence. Let $b_n(z)$ be a finite Blaschke product in the unit disk with zeros $\lambda_1,\ \lambda_2,\ \ldots \lambda_n$. We may suppose that $0 < |\lambda_1| < |\lambda_2| < \ldots < |\lambda_n|$. We try to find interpolating Blaschke product $B_m(z)$, $m \in N$, such that

$$B_m(z) \to b_n(z) \quad \text{if } m \to \infty$$

uniformly on the compact subsets of the unit disk U.

First, for $m \geq 2$, $x > 0$, define

$$R_k^{(m)}(x) = \frac{1}{m}\left(1 - \frac{1}{x \cdot m}\right)^k, \quad k = 1,\ 2,\ \ldots. \tag{1}$$

Note that

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} R_k^{(m)}(x) = x, \tag{2}$$

and

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} \left[R_k^{(m)}(x)\right]^2 = 0. \tag{3}$$

We put $\lambda_{n+1} = \lambda_1, \lambda_{n+2} = \lambda_2, \ldots, \lambda_{n+n} = \lambda_n\ \lambda_{n+k} = \lambda_k$, $k = 1,\ 2, \ldots$. Define

$$A_m = \sum_{k=n+1}^{\infty} \left[(1 - |\lambda_k|)^{\frac{1}{k}} - \frac{1}{\sqrt{m}}\right]^k.$$

Note that

$$0 < A_m < \left(1 - \frac{1}{\sqrt{m}}\right)^{n+1} \sqrt{m}\,.$$

Define $\lambda_k^{(m)} \in U$, $m \geq 2$, $k = 1,\ 2, \ldots$ so that

$$1 - \left|\lambda_k^{(m)}\right| = \left[(1 - |\lambda_k|)^{\frac{1}{k}} - \frac{1}{\sqrt{m}}\right]^k - R_k^{(m)}(A_m),$$

$$\arg \lambda_k^{(m)} = \arg \lambda_k, \quad k = 1, 2, \ldots, \tag{4}$$

where

$$A_m = \sum_{k=n+1}^{\infty} \left[(1 - |\lambda_k|)^{\frac{1}{k}} - \frac{1}{\sqrt{m}}\right]^k.$$

Note that $\lambda_j^{(m)} \neq \lambda_k^{(m)}$ if $j \neq k$ and $\lambda_k^{(m)} \to \lambda_k$, $m \to \infty$, for every k. According to (4) we have

$$\sum_{k=1}^{\infty}(1 - \left|\lambda_k^{(m)}\right|) = \sum_{k=1}^{n}\left[(1 - |\lambda_k|)^{\frac{1}{k}} - \frac{1}{\sqrt{m}}\right]^k$$

$$+A_m - \left(1 - \frac{1}{m}\right) A_m \to \sum_{k=1}^{n}(1 - |\lambda_k|)\,, \tag{5}$$

if $m \to \infty$. According to the definition of $\lambda_k^{(m)}$, we have that for every $m \geq 2$,

$$|\lambda_1| \leq |\lambda_k| < \left|\lambda_k^{(m)}\right|, \quad k = 1, 2, \ldots\,. \tag{6}$$

Let

$$c = \max_{1 \leq k \leq n-1}\left\{\frac{1 - |\lambda_{k+1}|}{1 - |\lambda_k|}\right\}. \tag{7}$$

Then (see for example [1], [2]), from (7) follows

$$\prod_{k,\, k \neq j}^{n}\left|\frac{\lambda_j - \lambda_k}{1 - \overline{\lambda}_j \lambda_k}\right| \geq \left(\prod_{k=1}^{n}\frac{1 - c^k}{1 + c^k}\right)^2 = \delta\,. \tag{8}$$

According to (8), (see for example [2]), we have

$$\sum_{k,\, k \neq j}^{n}\frac{\left(1 - |\lambda_j|^2\right)\,\left(1 - |\lambda_k|^2\right)}{|1 - \overline{\lambda}_j\,\lambda_k|^2} \leq 2\,\log\frac{1}{\delta} = C(\delta)\,. \tag{9}$$

Also, there is a δ_1 such that for all $k \neq j$

$$\left|\frac{\lambda_j - \lambda_j}{1 - \overline{\lambda}_j\,\lambda_k}\right| \geq \delta_1\,. \tag{10}$$

Now, we are able to prove that the sequence $\left(\lambda_k^{(m)}\right)_{k=1}^{\infty}$ is an interpolating sequence. Take

$$0 < \varepsilon$$

(11)

$$< \min\left\{\frac{(1 - |\lambda_n|^2)^2 C(\delta)}{n + (1 - |\lambda_n|^2) C(\delta)}, \frac{(1 - |\lambda_n|^2)^2 \delta_1}{8 - 4^{-1}\delta_1(1 - |\lambda_n|)^2}, \frac{(1 - |\lambda_n|^2)|\lambda_1|^2}{n}, \frac{1 - |\lambda_n|}{2}, \frac{|\lambda_n|}{2}\right\}.$$

There is a m_1 such that if $m \geq m_1$:

$$\left|\lambda_k^{(m)} - \lambda_k\right| < \frac{\varepsilon}{8}\,, \quad k = 1, 2, \dots\,. \tag{12}$$

Let

$$\begin{aligned} m_2 &= \frac{(1-|\lambda_1|^2)\left[n + (1-|\lambda_n|^2)\,C(\delta)\right]^2}{(1-|\lambda_n|^2)^2\left[n + (1-|\lambda_n|^2)^2 - 1\right]^2}\,,\ m_3 = C^{-2}(\delta), \\ m_4 &= \frac{4}{8(n+1)^2\varepsilon^2\left(\sqrt{2}+\sqrt{|\lambda_1|}\right)^2}, \end{aligned} \tag{13}$$

and choose $m_0 = \max\{m_1, m_2, m_3, m_4\}$. From (12) we have:

$$\begin{aligned} |1 - \overline{\lambda}_j^{(m)}\,\lambda_k^{(m)}|^2 &> |1 - \overline{\lambda}_j\lambda_k|^2 - 8\varepsilon, \\ |1 - \overline{\lambda}_j^{(m)}\,\lambda_k^{(m)}| &> |1 - \overline{\lambda}_j\lambda_k| - \frac{\varepsilon}{4}\,. \end{aligned} \tag{14}$$

Let $m \geq m_0$ and put for $j \neq k$

$$c_{j\,k} = \left|\frac{\lambda_j - \lambda_k}{1 - \overline{\lambda}_j\,\lambda_k}\right|\,, \quad c_{j\,k}^{(m)} = \left|\frac{\lambda_j^{(m)} - \lambda_k^{(m)}}{1 - \overline{\lambda}_j^{(m)}\,\lambda_k^{(m)}}\right|\,,$$

$$\tau_{j\,k} = \frac{(1-|\lambda_j|^2)(1-|\lambda_k|^2)}{|1 - \overline{\lambda}_j\,\lambda_k|^2}\,, \quad \tau_{j\,k}^{(m)} = \frac{(1-|\lambda_j^{(m)}|^2)(1-|\lambda_k^{(m)}|^2)}{|1 - \overline{\lambda}_j^{(m)}\,\lambda_k^{(m)}|^2}\,. \tag{15}$$

According to (12) and (6), we have

$$c_{jk}^{(m)} > c_{jk} - \frac{4\varepsilon}{(1-|\lambda_n|^2)(1-|\lambda_n|^2 - 4^{-1}\varepsilon)} = \delta_1 - \frac{1}{2}\delta_1 = \frac{\delta_1}{2} \tag{16}$$

$$\tau_{jk}^{(m)} \leq \tau_{jk} - \frac{\varepsilon}{[(1-|\lambda_n|^2)^2 - \varepsilon]\,(1-|\lambda_n|^2)^2} \leq C(\delta) + \frac{C(\delta)}{n}\,. \tag{17}$$

Furthermore, using (17) we have:

$$\sum_{k,k\neq j}\tau_{jk}^{(m)} = \sum_{k,k\neq j}^{n}\tau_{jk}^{(m)} + \sum_{n+1\geq k\neq j}\tau_{jk}^{(m)} \geq 2C(\delta)$$

$$+ \sum_{k\geq n+1}\frac{(1-|\lambda_1|^2)(1-|\lambda_k|^2)}{(1-|\lambda_n|^2)^2 - \varepsilon}$$

(18)

$$= 2C(\delta) + \frac{1-|\lambda_1|^2}{(1-|\lambda_n|^2)^2-\varepsilon}\left[A_m - \left(1-\frac{1}{m}\right)A_m\right] = 2C(\delta) + \frac{1}{\sqrt{m}} \leq 3C(\delta)\,.$$

Hence, by (14) and (16) it follows

(19)
$$\inf_j \prod_{k,\,k\neq j}\left|\frac{\lambda_j^{(m)}-\lambda_k^{(m)}}{1-\overline{\lambda}_j^{(m)}\lambda_k^{(m)}}\right| \geq e^{-12C(\delta)\,\delta_1^{-1}}\,.$$

By (19), the sequence $(\lambda_k^{(m)})_{k=1}^{\infty}$ is an interpolating sequence for every $m \geq m_0$.

Theorem 1. *Each finite Blaschke product in the unit disk can be approximated uniformly on the compact subsets of the unit disk by interpolating Blaschke product.*

Proof. Let $b_n(z)$ be the Blaschke product with zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$; $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|$. Suppose first that $n \geq 2$. Let $B_m(z)$, $m \geq m_0$, be the interpolating Blaschke products with zeros $\left(\lambda_k^{(m)}\right)_{k=1}^{\infty}$, $m \geq m_0$. It is easy to check that

(20)
$$\lim_{m\to\infty} b_n\left(\lambda_k^{(m)}\right) = 0,\ \lim_{m\to\infty} B_m(\lambda_k) = 0$$

for every k. Let $\varepsilon > 0$ be chosen as in (11). Put

(21)
$$\begin{aligned} c_m^{(n)}(z) &= \textstyle\prod_{k=1}^{n} \frac{\lambda_k^{(m)}-z}{1-\overline{\lambda}_k^{(m)}z}\,\frac{|\lambda_k^{(m)}|}{\lambda_k^{(m)}},\\ d_m^{(n)} &= \textstyle\prod_{k=n+1}^{\infty} \frac{\lambda_k^{(m)}-z}{1-\overline{\lambda}_k^{(m)}z}\,\frac{|\lambda_k^{(m)}|}{\lambda_k^{(m)}}\,. \end{aligned}$$

Similarly as in (18),

(22)
$$\lim_{m\to\infty} c_m^{(n)}(\lambda_k) = 0.$$

Let

(23)
$$r_m(z) = |b_n(z) - B_m(z)|\,.$$

If $z = \lambda_k$ or $z = \lambda_k^{(m)}$, then

(24)
$$\lim_{m\to\infty} r_m(\lambda_k) = 0 \ \text{ and } \ \lim_{m\to\infty} r_m(\lambda_k^{(m)}) = 0$$

for every k. Let $z \neq \lambda_k$, $z \neq \lambda_k^{(m)}$, $k = 1, 2, \ldots$, then we have

$$
\begin{aligned}
r_m(z) &= |b_n(z) - c_m^{(n)}(z) + c_m^{(n)} - B_n(z)| \\
&\leq |b_n(z) - c_m^{(n)}(z)| + |c_m^{(n)}(z) - c_m^{(n)}(z)\, d_m^{(n)}| \\
&\leq 2 \sum_{k=1}^{n} \frac{||l_k^{(m)}| - |\lambda_k|| + |\lambda_k|\lambda_k^{(m)} - |\lambda_k|\lambda_k^{(m)}}{(1-|z|)^2 |\lambda_k||\lambda_k^{(m)}|} \\
&+ 2|c_m^{(n)}(z)| \sum_{k=n+1}^{\infty} \frac{1-|\lambda_k^{(m)}|}{1-|z|} \leq 2 \sum_{k=1}^{n} \frac{|\lambda_k^{(m)}| - |\lambda_k|}{(1-|z|)^2 |\lambda_1|^2} \qquad (25)
\end{aligned}
$$

$$
+2|c_m^{(n)}(z)| \sum_{k=n+1}^{\infty} \frac{1-|\lambda_k^{(m)}|}{1-|z|} \leq 2 \sum_{k=1}^{n} \frac{|\lambda_k^{(m)}| - |\lambda_k|}{(1-|z|)^2 |\lambda_1|^2} + 2 \sum_{k=n+1}^{\infty} \frac{1-|\lambda_k|^{(m)}}{1-|z|} \to 0,
$$

if $m \to \infty$.

For example, if $n = 1$, then the theorem is true. If $b(z) = z$, then

$$
b(z) = \lim_{m\to\infty} \frac{z - \frac{1}{m}}{1 - \frac{1}{m} z} = \lim_{m\to\infty} B_m(z),
$$

where $B_m(z)$, $m \geq 2$, is a Blaschke product with zeros

$$
\lambda_k^{(m)} = 1 - \left[\left(1 - \frac{1}{m}\right)^{\frac{1}{k}} - \frac{1}{\sqrt{m}} \right]^k + \frac{1}{m} \left(1 - \frac{1}{mA_m}\right)^k, \quad k = 1, 2, \ldots \qquad (26)
$$

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