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A Conjecture about Positivity of the Polynomials Obtained by Expanding of a Product

Predrag M. Rajković [1], *Miloš Ivković* [2], *Marko Petković* [3]

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We try to make some contributions in proving the conjecture which P. Borwein established. In that order, we consider it in the matrix form and notice some wonderful relations. Also, we concentrate our attention to self-inversive polynomials and conclude that the whole conjecture can be written by three sequences of self-inversive polynomials. At last, our numerical evaluating persuade us that a few auxiliary conjectures are true We think that they can be useful in final proof.

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1. Introduction

Let us denote by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$$

and

$$J(n;q) = (q;q^3)_n\ (q^2;q^3)_n = \prod_{i=0}^{n-1}(1-q^{3i+1})(1-q^{3i+2}).$$

For fixed n, $J(n;q)$ is a monotonously decreasing function in the interval $[0,1]$ with a unique point of inflection $\xi \approx 0.448527$.

Peter Borwein in 1990 made the next conjecture.

Conjecture 1.1. (**P. Borwein**) *The polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ defined by*

$$J(n;q) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3), \tag{1.1}$$

have nonnegative coefficients.

Some similar conjectures are discussed by D.M. Bressoud [3].

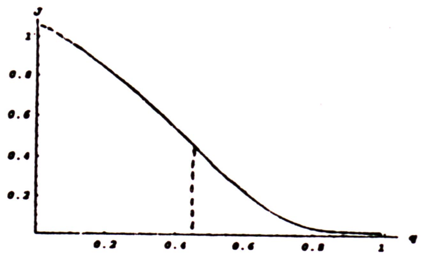

Figure 1.a

The function $J(n;q)$ for $n=7$.

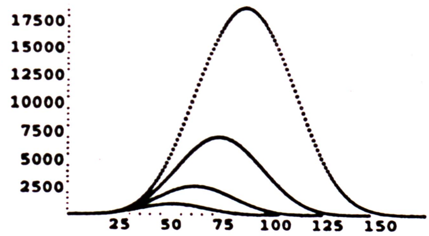

Figure 1.b

The coefficients of $A_n(q)$ $(n=10,11,12,13)$.

In the paper [2], G.E. Andrews has derived the next recurrence relations

$$A_n(q)=(1+q^{2n-1})A_{n-1}(q)+q^nB_{n-1}(q)+q^nC_{n-1}(q),\quad A_0(q)=1,$$

$$B_n(q)=q^{n-1}A_{n-1}(q)+(1+q^{2n-1})B_{n-1}(q)-q^nC_{n-1}(q),\quad B_0(q)=0,$$

$$C_n(q)=q^{n-1}A_{n-1}(q)-q^{n-1}B_{n-1}(q)+(1+q^{2n-1})C_{n-1}(q),\quad C_0(q)=0.$$

Notice that

$$\deg A_n(q)=n^2,\quad \deg B_n(q)=\deg C_n(q)=n^2-1\quad (n>0).$$

If we write the Andrews' recurrence relations in the form

$$A_n(q)-A_{n-1}(q)=q^n\left\{q^{n-1}A_{n-1}(q)+B_{n-1}(q)+C_{n-1}(q)\right\},$$

$$B_n(q)-B_{n-1}(q)=q^{n-1}\left\{A_{n-1}(q)+q^nB_{n-1}(q)-qC_{n-1}(q)\right\},$$

$$C_n(q)-C_{n-1}(q)=q^{n-1}\left\{A_{n-1}(q)-B_{n-1}(q)+q^nC_{n-1}(q)\right\},$$

we see that the polynomials $A_n(q)=\sum_{j=0}^{n^2}a_{n,j}q^j$, $B_n(q)=\sum_{j=0}^{n^2-1}b_{n,j}q^j$ and $C_n(q)=\sum_{j=0}^{n^2-1}c_{n,j}q^j$, have the property

$$a_{n,i}=a_{n-1,i},\quad b_{n,i}=b_{n-1,i},\quad c_{n,i}=c_{n-1,i}\quad (i=0,\ldots,n-2).$$

Moreover $a_{n,n-1}=a_{n-1,n-1}$.

According to Figure 1.b, we can believe that the following is true.

Conjecture 1.2. *The coefficients $a_{n,i}$ are increasing functions of index n, i.e.,*

$$a_{n,i} \leq a_{n+1,i} \quad (i = 1, 2, \ldots, [n^2/2]) \quad (n = 1, 2, \ldots).$$

Starting from $n = 7$, the coefficients $a_{n,i}$ are increasing by index i, i.e.,

$$a_{n,i-1} \leq a_{n,i} \quad (i = 1, 2, \ldots, [n^2/2]) \qquad (n = 7, 8, \ldots).$$

The Andrews' recurrence relations can be written in the next matrix form

$$X_n(q) = F(n, q)X_{n-1} \quad (n = 1, 2, \ldots),$$

where

$$X_n(q) = \begin{bmatrix} A_n(q) \\ B_n(q) \\ C_n(q) \end{bmatrix}, \quad F(n, q) = \begin{bmatrix} (1 + q^{2n-1}) & q^n & q^n \\ q^{n-1} & (1 + q^{2n-1}) & -q^n \\ q^{n-1} & -q^{n-1} & (1 + q^{2n-1}) \end{bmatrix}.$$

It can be of interest that

$$\det F(n, q) = 1 + (q^3)^{2n-1} - q(q^3)^{n-1} - q^2(q^3)^{n-1}$$

has the similar expanding as $J(n; q)$.

Lemma 1.1. *The matrices $\{F(n, q)\}$ are commutative, i.e.*

$$F[k, q]F[n, q] = F[n, q]F[k, q].$$

Lemma 1.2. *The recurrence relations can be written in the form*

$$\begin{bmatrix} A_n & qC_n & qB_n \\ B_n & A_n & -qC_n \\ C_n & -B_n & A_n \end{bmatrix}$$

$$= \begin{bmatrix} (1 + q^{2n-1}) & q^n & q^n \\ q^{n-1} & (1 + q^{2n-1}) & -q^n \\ q^{n-1} & -q^{n-1} & (1 + q^{2n-1}) \end{bmatrix} \begin{bmatrix} A_{n-1} & qC_{n-1} & qB_{n-1} \\ B_{n-1} & A_{n-1} & -qC_{n-1} \\ C_{n-1} & -B_{n-1} & A_{n-1} \end{bmatrix}.$$

Lemma 1.3. *The matrix $F(n, q)$ can be decomposed as follows:*

$$F(n, q) = q^{n-1}L + (1 + q^{2n-1})I + q^n L^t, \qquad \textit{where} \qquad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

and L^t is transpose matrix of L.

Obviously,

$$L^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad (L^t)^2 = (L^2)^t \qquad L^n = (L^t)^n = 0 \quad (n = 3, 4 \ldots).$$

By mathematical induction, we can prove the next lemma.

Lemma 1.4. *It is valid*

$$(LL^t)^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & f_{2n-2} & f_{2n-1} \\ 0 & f_{2n-1} & f_{2n} \end{bmatrix}, \quad (L^tL)^n = \begin{bmatrix} f_{2n} & -f_{2n-1} & 0 \\ -f_{2n-1} & f_{2n-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (n > 0),$$

where f_n are the Fibonacci numbers:

$$f_0 = f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad (n = 2, 3, \ldots).$$

2. The Borwein conjecture through eigenvalues

The matrix $F(n, q)$ is of great importance in the Borwein conjecture. So, let us examine it by spectral analysis.

Theorem 2.1. *The matrix $F(n, q)$ has the eigenvalues*

$$\lambda_1(n; q) = (1 - q^{n-2/3})(1 - q^{n-1/3})$$

$$\lambda_2(n; q) = (1 + \frac{1}{2}q^{n-2/3} + \frac{1}{2}q^{n-1/3} + q^{2n-1}) + i\frac{\sqrt{3}}{2}q^{n-2/3}(1 - q^{1/3})$$

$$\lambda_3(n; q) = (1 + \frac{1}{2}q^{n-2/3} + \frac{1}{2}q^{n-1/3} + q^{2n-1}) - i\frac{\sqrt{3}}{2}q^{n-2/3}(1 - q^{1/3}).$$

These eigenvalues have the following properties:

$$J(n; q) = \prod_{k=1}^{n} \lambda_1(k; q^3),$$

$$\lambda_1(n; q^3) = \lambda_1(n; q) \cdot \lambda_2(n; q) \cdot \lambda_3(n; q),$$

$$\lambda_2(n; q) \cdot \lambda_3(n; q) = (1 + q^{n-1/3} + q^{2n-2/3})(1 + q^{n-2/3} + q^{2n-4/3}).$$

Theorem 2.2. *The matrix $F(n,q)$ can be decomposed by*

$$F[n,q] = M \cdot D_n \cdot M^{-1},$$

where the matrix $D_n = \operatorname{diag}\{\lambda_1(n;q), \lambda_2(n;q), \lambda_3(n;q)\}$ and the matrices M and M^{-1} do not depend on index n and they are given by:

$$M = \begin{bmatrix} -q^{2/3} & \frac{1-i\sqrt{3}}{2}q^{2/3} & \frac{1+i\sqrt{3}}{2}q^{2/3} \\ q^{1/3} & -\frac{1+i\sqrt{3}}{2}q^{1/3} & \frac{-1+i\sqrt{3}}{2}q^{1/3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{3q^{2/3}} \begin{bmatrix} -1 & q^{1/3} & q^{2/3} \\ \frac{1+i\sqrt{3}}{2} & \frac{1-i\sqrt{3}}{2}q^{1/3} & q^{2/3} \\ \frac{1-i\sqrt{3}}{2} & -\frac{1+i\sqrt{3}}{2}q^{1/3} & q^{2/3} \end{bmatrix}.$$

Theorem 2.3. *It is valid*

$$\prod_{i=1}^{n} F[i,q] = M \cdot \Big(\prod_{i=1}^{n} D_i\Big) \cdot M^{-1}.$$

Theorem 2.4. *The polynomial $A_n(q)$ can be expressed by*

$$A_n(q) = \frac{1}{3}\left\{\prod_{k=1}^{n} \lambda_1(k;q) + \prod_{k=1}^{n} \lambda_2(k;q) + \prod_{k=1}^{n} \lambda_3(k;q)\right\}.$$

3. Some reciprocal polynomials in the conjecture

We remind that a polynomial $A(q) = a_n q^n + a_{n-1}q^{n-1} + \cdots + a_0$ is *reciprocal* if $a_{n-k} = a_k$ $(k = 0,1,\ldots,n)$, i.e., $A(q) = q^n A(1/q)$.

Yet G.E. Andrews [2] has noticed that

$$C_n(q) = q^{n^2-1} B_n(1/q) \quad (n \in \mathbf{N}), \tag{3.1}$$

i.e. that $B_n(q)$ and $C_n(q)$ are inversive to each other. Hence

$$c_{n,i} = b_{n,n^2-1-i} \quad (i = 0,1\ldots,n^2-1).$$

We can notice some sequences of reciprocal polynomials.

Let us denote by

$$D_n(q) = B_n(q) + C_n(q), \qquad E_n(q) = B_n(q) + qC_n(q).$$

The polynomials $A_n(q)$, $D_n(q)$ and $E_n(q)$ reciprocal, i.e.,

$$A_n(q) = q^{n^2} A_n(1/q), \quad D_n(q) = q^{n^2-1} D_n(1/q), \quad E_n(q) = q^{n^2-1} E_n(1/q).$$

The sequences $\{A_n(q)\}$, $\{D_n(q)\}$ and $\{E_n(q)\}$ satisfy the next recurrence relations

$$\begin{bmatrix} A_n \\ D_n \\ E_n \end{bmatrix} = \begin{bmatrix} 1+q^{2n-1} & q^n & 0 \\ 2q^{n-1} & 1+q^{2n-1} & -q^{n-1} \\ q^{n-1}+q^n & -q^n & 1+q^{2n-1} \end{bmatrix} \begin{bmatrix} A_{n-1} \\ D_{n-1} \\ E_{n-1} \end{bmatrix}, \quad \begin{bmatrix} A_0 \\ D_0 \\ E_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, introducing the reciprocal polynomial

$$F_n = A_n + E_n,$$

we can prove:

The sequences $\{A_n(q)\}$, $\{D_n(q)\}$ and $\{F_n(q)\}$ satisfy the next recurrence relations

$$\begin{bmatrix} A_n \\ D_n \\ F_n \end{bmatrix} = \begin{bmatrix} 1+q^{2n-1} & q^n & 0 \\ 3q^{n-1} & 1+q^{2n-1} & -q^{n-1} \\ q^{n-1}+q^n & 0 & 1+q^{2n-1} \end{bmatrix} \begin{bmatrix} A_{n-1} \\ D_{n-1} \\ F_{n-1} \end{bmatrix}, \quad \begin{bmatrix} A_0 \\ D_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

4. The fundamental recurrence relation

It seems to be of great importance to find separate recurrence relations for the sequences $\{A_n(q)\}$, $\{B_n(q)\}$ and $\{C_n(q)\}$.

Theorem 4.1. *The sequences $\{A_n(q)\}$, $\{B_n(q)\}$ and $\{C_n(q)\}$ satisfy the same recurrence relation*

$$\mathcal{F}_{n+3} = \mathcal{F}_{n+2} \cdot (1+q+q^2)(1+q^{2n+3})$$

$$-\mathcal{F}_{n+1} \cdot q(1+q+q^2)(1+q^{2n+2}+q^{4n+4}) + \mathcal{F}_n \cdot q^3(1-q^{3n+1})(1-q^{3n+2}),$$

with only difference in initial values:

$$\mathcal{F}_n = A_n(q): \quad A_0 = 1, \quad A_1 = 1+q, \quad A_2(q) = 1+q+2q^2+q^3+q^4,$$

$$\mathcal{F}_n = B_n(q): \quad B_0 = 0, \quad B_1 = 1, \quad B_2(q) = 1+q+q^3,$$

$$\mathcal{F}_n = C_n(q): \quad C_0 = 0, \quad C_1 = 1, \quad C_2(q) = 1 + q^2 + q^3,$$
$$\mathcal{F}_n = D_n(q): \quad D_0 = 0, \quad D_1 = 2, \quad D_2(q) = 2 + q + q^2 + 2q^3,$$
$$\mathcal{F}_n = E_n(q): \quad E_0 = 0, \quad E_1 = 1 + q, \quad E_2(q) = 1 + 2q + 2q^3 + q^4,$$
$$\mathcal{F}_n = F_n(q): \quad F_0 = 1, \quad F_1 = 2 + 2q, \quad F_2(q) = 2 + 3q + 2q^2 + 3q^3 + 2q^4.$$

Proof. The most simple way to find the recurrence relation is from the relations for the sequences $\{A_n(q)\}$, $\{D_n(q)\}$ and $\{F_n(q)\}$ which are given in Theorem 3.3. From the second relation we have

$$q^{n-1}F_{n-1} = 3q^{n-1}A_{n-1} + (1 + q^{2n-1})D_{n-1} - D_n \quad (n \in \mathbb{N}).$$

Now, we can eliminate F_{n-1} and F_n from the third relation, i.e. we have

$$3q^n A_n - (q^{2n-1}(1+q) + 3q^n(1 + q^{2n-1}))A_{n-1}$$
$$= D_{n+1} - (1+q)(1+q^{2n})D_{n-1} + q(1+q^{2n-1})^2 D_{n-1}.$$

From the first relation of Theorem 3.3, we have

$$q^n D_{n-1} = A_n - (1 + q^{2n-1})A_{n-1} \quad (n \in \mathbb{N}).$$

We use it for D_{n-1}, D_n and D_{n+1} and put in the previous relation. After changing $n \to n+1$, we get the wanted difference equation. ■

The recurrence relation for $A_n(q)$ can be written in the form

$$A_{n+3}(q) = (1+q+q^2)\left\{(1+q^{2n+3})A_{n+2}(q) - q(1+q^{2n+2})^2 A_{n+1}(q)\right\}$$
$$+q^3\left\{q^{2n}(1+q+q^2)A_{n+1}(q) + (1-q^{3n+1})(1-q^{3n+2})A_n(q)\right\}.$$

Conjecture 4.2. *The polynomials*

$$(1+q^{2n+3})A_{n+2}(q) - q(1+q^{2n+2})^2 A_{n+1}(q)$$

and

$$q^{2n}(1+q+q^2)A_{n+1}(q) + (1-q^{3n+1})(1-q^{3n+2})A_n(q)$$

are polynomials with non-negative coefficients.

5. The zeros

May be it can be useful to examine the zeros of polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$.

Lemma 5.1. *The polynomial $A_{2n+1}(q)$ vanishes in the point -1. Moreover, it can be written in the form*

$$A_{2n+1}(q) = (1+q)\hat{A}_{2n+1}(q),$$

where $\hat{A}_{2n+1}(q)$ is a self-inversive polynomial.

Proof. From recurrence relation for the sequence $\{A_n(q)\}$, taking $q = -1$, we obtain $A_{2n+3}(-1) = 3A_{2n+1}(-1)$. Since $A_1(-1) = 0$, by mathematical induction, we have $A_{2n+1}(-1) = 0$ for all $n \in \mathbb{N}$. Hence, we can write $A_{2n+1}(q) = (1+q)\hat{A}_{2n+1}(q)$, where $\hat{A}_{2n+1}(q)$ is a polynomial of degree $(2n+1)^2 - 1$. Knowing that $A_{2n+1}(q)$ is the self-inversive polynomial,i.e. $A_{2n+1}(q) = q^{(2n+1)^2} A_{2n+1}(1/q)$, by simple change, we have

$$\hat{A}_{2n+1}(q) = q^{(2n+1)^2-1}\hat{A}_{2n+1}(1/q).$$

■

Conjecture 5.1. *The polynomial $\hat{A}_{2n+1}(q)$ has all positive coefficients.*

Lemma 5.2. *If the polynomial $A_n(q)$ vanishes at the point $z = Re^{it}$, then it also vanishes at the points $z = Re^{-it}$, $z = e^{it}/R$ and $z = e^{-it}/R$.*

Proof. Since $A_n(q)$ has all real coefficients, it follows that the complex zeros appear in conjugate pairs. From the fact that $A_n(q)$ is self-inversive polynomial we conclude that, if $z = Re^{it}$ is a zero, then also $1/z = e^{-it}/R$. ■

Our numerical evaluating persuade us that all the zeros of $A_n(q)$ lie in the ring $1 - \varepsilon < |z| < 1 + \varepsilon$, where $0 < \varepsilon < 1/2$.

Especially, when we consider a zero $z = Re^{it}$ $(R \neq 0)$, according to Lemma 8.2, we have 4 different zeros and the next product

$$(z - Re^{it})(z - Re^{-it})(z - e^{it}/R)(z - e^{-it}/R)$$

$$= z^4 - 2(R + \frac{1}{R})\cos t \cdot z(1+z^2) + \left\{(R + \frac{1}{R})^2 + 2\cos(2t)\right\} z^2 + 1$$

in the polynomial $A_n(q)$.

In the case $R = 1$, this product can be written like $(z^2 - 2z\cos t + 1)^2$. But, only one or two zeros appear from this four point set.

Similar behavior we notice for the zeros of $B_n(q)$ and $C_n(q)$.

6. No doubt, the positive sequences

The sequences of polynomials $\{A_n^+(q)\}$, $\{B_n^+(q)\}$ and $\{C_n^+(q)\}$ derived from the relation

$$\prod_{i=1}^{n}(1+q^{3i-2})(1+q^{3i-1}) = A_n^+(q^3) + qB_n^+(q^3) + q^2C_n^+(q^3),$$

are, no doubt, with positive coefficients.

Remembering the Borwein conjecture

$$\prod_{i=1}^{n}(1-q^{3i-2})(1-q^{3i-1}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3),$$

and multiplying the same sides of equalities, we can prove the next theorem.

Theorem 6.1 *The sequences $\{A_n(q)\}$, $\{B_n(q)\}$ and $\{C_n(q)\}$ are connected with the sequences $\{A_n^+(q)\}$, $\{B_n^+(q)\}$ and $\{C_n^+(q)\}$, by the relations:*

$$A_n(q^2) = A_n^+(q)A_n(q) - qC_n^+(q)B_n(q) - qB_n^+(q)C_n(q),$$

$$B_n(q^2) = -C_n^+(q)A_n(q) + B_n^+(q)B_n(q) + A_n^+(q)C_n(q),$$

$$qC_n(q^2) = -B_n^+(q)A_n(q) + A_n^+(q)B_n(q) + qC_n^+(q)C_n(q).$$

7. The generating functions and Laplace transform

Denote by

$$\alpha(z) = \sum_{n=0}^{\infty} A_n(q)z^n, \quad \beta(z) = \sum_{n=0}^{\infty} B_n(q)z^n, \quad \gamma(z) = \sum_{n=0}^{\infty} C_n(q)z^n$$

the generating functions of the sequences of polynomials included in the Borwein conjecture.

Theorem 7.1. *The generating functions $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ satisfy the next system of the functional equations*

$$z^{-1}[\alpha(z) - 1] = \alpha(z) + q\alpha(q^2z) + q\beta(qz) + q\gamma(qz),$$

$$z^{-1}\beta(z) = \alpha(qz) + \beta(z) + q\beta(q^2z) - q\gamma(qz),$$

$$z^{-1}\gamma(z) = \alpha(qz) - \beta(qz) + \gamma(z) + q\gamma(q^2z),$$

with initial values

$$\alpha(0) = A_0 = 1, \quad \beta(0) = B_0 = 0, \quad \gamma(0) = C_0 = 0.$$

Using the fundamental recurrence relation, we prove

Theorem 7.2. *The generating function* $\alpha(z)$ *satisfies the next functional equation*

$$q^7z^3\ \alpha(q^6z) - (1+q+q^2)\left\{(qz)^2\ \alpha(q^4z) + q^4z^3\ \alpha(q^3z) - (1-qz)z\ \alpha(q^2z)\right\}$$

$$-q(1-z)(1-qz)(1-q^2z)\ \alpha(z)+q[(1-q-2q^2-q^3+q^4+q^5)z+1+2(1+q+q^2)] = 0.$$

Let us apply the Laplace transform on the sequences $\{A_n(q)\}$, $\{B_n(q)\}$ and $\{C_n(q)\}$. Denote by

$$\mathcal{L}[A_n(q)] = a_n(p), \quad \mathcal{L}[B_n(q)] = b_n(p), \quad \mathcal{L}[C_n(q)] = c_n(p).$$

Knowing that

$$\mathcal{L}[q^k] = \frac{k!}{q^{k+1}}, \qquad \mathcal{L}[q^kA_n(q)] = (-1)^k a_n^{(k)}(p),$$

we have:

Theorem 7.3. *The Borwein conjecture can be written in the form*

$$a_n(p) = a_{n-1}(p) - a_{n-1}^{(2n-1)}(p) + (-1)^n b_{n-1}^{(n)}(p) + (-1)^n c_{n-1}^{(n)}(p),$$

$$b_n(p) = (-1)^{n-1} a_{n-1}^{(n-1)}(p) + b_{n-1}(p) - b_{n-1}^{(2n-1)}(p) - (-1)^n c_{n-1}^{(n)}(p),$$

$$c_n(p) = (-1)^{n-1} a_{n-1}^{(n-1)}(p) - (-1)^{n-1} b_{n-1}^{(n-1)}(p) + c_{n-1}(p) - c_{n-1}^{(2n-1)}(p).$$

Here, the improvement is in the fact that we find recurrence relations with constant coefficients.

8. Conjectures for further research

By using package *Mathematica*, we have done a lot of trials which persuade us that the next statements are true.

Conjecture 8.1. *The polynomial* $B_n(q) = \sum_{j=0}^{n^2-1} b_{n,j}q^j$ *has the property*

$$b_{n,n^2-2} = 0, \quad b_{n,j} > 0, \ j \neq n^2 - 2.$$

Equivalently, knowing $C_n(q) = q^{n^2-1}B_n(1/q) \quad (n \in \mathbf{N})$, we can establish the next conjecture.

Conjecture 8.2. *The polynomial* $C_n(q) = \sum_{j=0}^{n^2-1} c_{n,j}q^j$ *has the property*

$$c_{n,1} = 0, \quad c_{n,j} > 0, \ j \neq 1.$$

Conjecture 8.3. *The coefficients of the polynomials* $A_n(q) - B_n(q)$ *and* $A_n(q) - qB_n(q)$ *are positive.*

Conjecture 8.4. *The coefficients of the polynomial* $A_n(q) - C_n(q)$ *and* $A_n(q) - qC_n(q)$ *are positive.*

Conjecture 8.5. *The coefficients of the polynomials* $A_n(q) - (1 + q^{2n-1})A_{n-1}(q)$, $B_n(q) - (1 + q^{2n-1})B_{n-1}(q)$ *and* $C_n(q) - (1 + q^{2n-1})C_{n-1}(q)$ *are positive.*

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[1] *Department of Mathematics*
Faculty of Mechanical Engineering
University of Niš
Niš , SERBIA and MONTENEGRO
e-mail: pecar@masfak.ni.ac.yu

[2] *IMECC-UNICAMP, C.P. 6065*
13083-970 Campinas-SP, BRAZIL
e-mail: milos@ime.unicamp.br

[3] *Faculty of Electronic Engineering*
University of Niš
Niš , SERBIA and MONTENEGRO

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