Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 3-4

Effect of Junction on Electromagnetic Waveguide Transmission

Toni Sayah [1] *and Badih Azar* [2]

Presented at Internat. Congress "MASSEE' 2003", 4th Symposium "TMSF"

In [5], we couple finite element method and modal expansion method to study electromagnetic waveguides junctions. In this paper, we use the same approach to calculate the reflected and transmitted parts of the modes. We introduce a variational formulation and show numerical result for a simple case. Numerical complicated results will be showed later.

AMS Subj. Classification: 78A50, 78M10, 78M30

Key Words: waveguide, propagated modes, finite elements, impedance condition

1. Introduction

Junctions has a principal effect on the electromagnetic propagation by waveguides. The form and the material of the junction characterize the transmission of the wave. A lot of authors had studied this problem with different technics, Sieverding and Arndt [7], Taysh and Butler [9] and [8], V.N. Kanelloupoulos and J.P.Webb [3], M.D. Deshpande, C.J. Reddy and M.C. Bailey [1], E. Limit, E. Martini, G. Pelosi, M. Pierozzi and S. Selleri [4]. G. Shen, Look and Qian treat a particular case, the magic waveguide T-Junction [6]. In [5], we study the junction of a circular waveguide. We introduce an impedance condition using Dirichlet-Neumann operator which allows us to introduce a variational formulation. Using the finite element method, we show numerical result in a simple case. This approach works as well as the junction is matched and there is no reflected part.

In this paper, we use the technics introduced in [5], but we suppose knowing the incident wave and we calculate the transmitted and reflected coefficients. Finally, we introduce the corresponding variational formulation and the numerical system and we show numerical result in a simple case. This work will be continued later to show numerical results in complicated cases.

2. Statement of the problem

We consider a junction $\Omega^j \in \mathbb{R}^3$ of surface Γ^j between three semi-infinite circular waveguides (C0,C1,C2) constituted by an isolating domain Ω^i of surface Γ^i and diameter $d = 2a$. We denote by ν the exterior unitary normal on the surfaces.

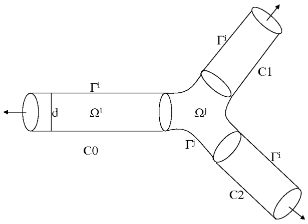


Figure 1: Geometry

For simplicity, we suppose that the permeability μ and the permittivity ϵ are constants in $\Omega^i \cup \Omega^j$ and the surface $\Gamma^i \cup \Gamma^j$ is perfectly conductive. We denote by f the wave frequency, $\omega = 2\pi f$ the pulsation and $k = \omega\sqrt{\mu\epsilon}$ the wave number.
In $\Omega^i \cup \Omega^j$, the electromagnetic field verifies Maxwell equations in harmonic regime:

$$\begin{cases} CurlE - iw\mu H = 0 \\ CurlH + iw\epsilon E = 0 \\ divE = 0 \quad ; \quad divH = 0 \end{cases}$$

with the boundary conditions on $\Gamma^i \cup \Gamma^j$: $E \wedge \nu = 0$ and $H \cdot \nu = 0$.

We suppose that an electromagnetic wave (E^i, H^i) enter by C0. Our goal is to study the effect of the junction on this wave and to calculate the reflected (E^d, H^d) part on C0 and transmitted parts across C1 and C2. In C0 we have : $E = E^i + E^d$ and $H = H^i + H^d$.

3. Infinite waveguide

In [5], we studied the infinite circular waveguide and we introduced an impedance condition on a cross surface. In this section, we recall the Dirichlet-Neumann operator introduced in [5].

We work on cylindrical coordinates (ρ, ϕ, z). For a circular waveguide with axis z, the propagated modes are the transverse electric TE and the transverse magnetic TM [2] ($n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+_*$) :

$$TE_{nm}\begin{cases} (E^{TE}_{nm})_{\|} = a^{TE}_{nm}\left(\frac{in}{\rho}J_n(k^1_{nm}\rho)e_\rho - k^1_{nm}J^{'}_n(k^1_{nm}\rho)e_\phi\right)g^1_{nm} \\ (H^{TE}_{nm})_{\|} = \frac{a^{TE}_{nm}}{i\omega\mu}\left(i\beta^1_{nm}k^1_{nm}J^{'}_n(k^1_{nm}\rho)e_\rho - \frac{n\beta^1_{nm}}{\rho}J_n(k^1_{nm}\rho)\right)g^1_{nm} \end{cases}$$

$$TM_{nm}\begin{cases} (H^{TM}_{nm})_{\|} = a^{TM}_{nm}\left(\frac{in}{\rho}J_n(k^2_{nm}\rho)e_\rho - k^2_{nm}J^{'}_n(k^2_{nm}\rho)e_\phi\right)g^2_{nm} \\ (E^{TM}_{nm})_{\|} = -\frac{a^{TM}_{nm}}{i\omega\varepsilon}\left(i\beta^2_{nm}k^2_{nm}J^{'}_n(k^2_{nm}\rho)e_\rho - \frac{n\beta^2_{nm}}{\rho}J_n(k^2_{nm}\rho)\right)g^2_{nm}, \end{cases}$$

where (e_ρ, e_ϕ, e_z) are the unit vectors of the cylindrical axis, $V_{\|} = e_z \wedge V \wedge e_z$, a^{TE}_{nm} and a^{TM}_{nm} are constants defining the modes, ak^1_{nm} (resp. ak^2_{nm}) is the mth zeros of the Bessel functions of first kind J_n (resp. $J^{'}_n$), $\beta^1_{nm} = \sqrt{k^2 - (k^1_{nm})^2}$ and $\beta^2_{nm} = \sqrt{k^2 - (k^2_{nm})^2}$, $g^1_{nm} = e^{in\phi}e^{i\beta^1_{nm}z}$ and $g^2_{nm} = e^{in\phi}e^{i\beta^2_{nm}z}$.

We denote by f^{TE}_{nm} (resp. f^{TM}_{nm}) the cutoff frequency of the general mode TE_{nm} (resp. TM_{nm}). Then, if $f < f^{TE}_{nm}$, this mode will not propagate. The cutoff frequencies of the first mode TE_{11} and the second mode TM_{01} are respectively $f^{TE}_{11} = 87.843/a \quad GHZ$ and $f^{TM}_{01} = 114.754/a \quad GHZ$ where a is in mm.

The electromagnetic field (E, H) in the waveguide can be written as

$$\text{(CH)}\begin{cases} E(\rho,\phi,z) &= \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+_*}\left(E^{TE}_{nm} + E^{TM}_{nm}\right) \\ H(\rho,\phi,z) &= \sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+_*}\left(H^{TE}_{nm} + H^{TM}_{nm}\right) \end{cases}$$

We deduce the relation $(CurlE)_{\|} = i\omega\mu H_{\|} = R(e_z \wedge E)$, where R, introduced in [5], is called the Dirichlet-Neumann operator.

4. Problem in bounded domain

The Dirichlet-Neumann operator allows us to solve the problem in bounded domain obtained by cutting the circular waveguides by cross sections. Let Ω be a domain of surface $\partial\Omega = \Gamma \cup \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ and ν the exterior unitary normal on $\partial\Omega$. We denote by R_0, R_1 and R_2 the Dirichlet-Neumann operators on the

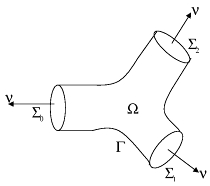


Figure 2: Bounded domain

surfaces Σ_0, Σ_1 and Σ_2. Then E verifies the system :

$$(P)\left\{\begin{array}{ll} Curl\ \dfrac{1}{\mu}Curl\ E - k^2\varepsilon E = 0 & \text{in } \Omega \\ E\wedge n = 0 & \text{on } \Gamma \\ \nu\wedge E(\rho,\phi,z) = \nu\wedge E^i(\rho,\phi,z) & \\ \qquad\qquad + \displaystyle\sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+_*} M^{0,\rho}_{n,m}\, e_\rho + M^{0,\phi}_{n,m}\, e_\phi & \text{on } \Sigma_0 \\ (Curl\ E)_{\|} - R_0(\nu\wedge E) = (Curl\ E^i)_{\|} - R_i(\nu\wedge E^i) & \text{on } \Sigma_0 \\ \text{for} \quad l=1,2 \quad we\ \ have\ : & \\ \quad \nu\wedge E(\rho,\phi,z) = \displaystyle\sum_{n\in\mathbb{Z}^+}\sum_{m\in\mathbb{Z}^+_*} M^{l,\rho}_{n,m}\, e_\rho + M^{l,\phi}_{n,m}\, e_\phi & \text{on } \Sigma_l \\ \quad (Curl\ E)_{\|} - R_l(\nu\wedge E) = 0 & \text{on } \Sigma_l \end{array}\right.$$

5. Variational formulation

In this section, we show a variational formulation corresponding to the problem (P). First, we introduce the space V where we search the field E

$$V = \{u\in L^2(\Omega)^3, \quad Curl\ u\in L^2(\Omega)^3,\ \nu\wedge u = 0 \ \text{ on } \ \Gamma\}$$

For F in V we have

$$\int_\Omega CurlE\cdot Curl\ \bar{F}d\Omega - k^2\int_\Omega E\cdot\bar{F}d\Omega - \int_{\partial\Omega} Curl\ E\cdot(\nu\wedge\bar{F})d\sigma = 0$$

with

$$\begin{aligned} \int_{\partial\Omega} Curl\ E\cdot(\nu\wedge\bar{F})d\sigma \ &= \ \int_{\Sigma_1} R_1(\nu\wedge E)\cdot(\nu\wedge\bar{F})d\sigma + \int_{\Sigma_2} R_2(\nu\wedge E)\cdot(\nu\wedge\bar{F})d\sigma \\ &+ \int_{\Sigma_0} R_0(\nu\wedge E - \nu\wedge E^i)\cdot(\nu\wedge\bar{F})d\sigma + \int_{\Sigma_0} Curl\ E^i\cdot(\nu\wedge\bar{F})d\sigma \end{aligned}$$

Then the corresponding variational formulation is

$$(V)\left\{\begin{array}{l}\text{find E} \in V \text{ such that , } \forall F \in V \text{ we have} \\ \displaystyle\int_\Omega Curl\ E \cdot Curl\ \bar{F} d\Omega - k^2 \int_\Omega E \cdot \bar{F} d\Omega\ - \\ \displaystyle\int_{\Sigma_1} R_1(\nu \wedge E) \cdot (\nu \wedge \bar{F}) d\sigma + \int_{\Sigma_2} R_2(\nu \wedge E) \cdot (\nu \wedge \bar{F}) d\sigma \\ +\int_{\Sigma_0} R_0(\nu \wedge E - \nu \wedge E^i) \cdot (\nu \wedge \bar{F}) d\sigma = -\displaystyle\int_{\Sigma_0} Curl\ E^i \cdot (\nu \wedge \bar{F}) d\sigma \end{array}\right.$$

6. Approximation

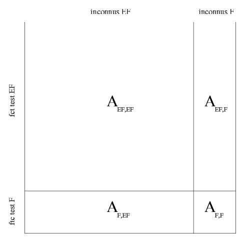
In fact, for a frequency f, we have a finite number of propagated modes. For the approximation, we can take these propagated modes and some evanescent modes which has a non neglected effect. The modes which have the cutoff frequency very far of f will be neglected. Then the infinite double summation $\sum_{n \in \mathbb{Z}^+} \sum_{m \in \mathbb{Z}^+_*}$ can be approximated by a finite summation $\sum_{p=0}^{N} \sum_{q=1}^{M}$ For the discretisation, we use the H_{rot} volume finite elements coupled with the modes. To simplify the numerical method, we introduce auxiliary variables $(a_{nm}^{0,TE}, a_{nm}^{0,TM})$, $(a_{nm}^{1,TE}, a_{nm}^{1,TM})$ and $(a_{nm}^{2,TE}, a_{nm}^{2,TM})$ which are the Fourier coefficients of $\nu \wedge E^d(\rho, \phi, z)$ on Σ_0 and $M = \nu \wedge E(\rho, \phi, z)$ on Σ_1 and Σ_2. Using the formula (CH) we write

$$\left\{\begin{array}{ll} \nu \wedge (E(\rho,\phi,z) - E^i(\rho,\phi,z)) = \displaystyle\sum_{n \in \mathbb{Z}^+} \sum_{m \in \mathbb{Z}^+_*} \left(a_{nm}^{0,TE}\, V_{nm} + a_{nm}^{0,TM}\, W_{nm}\right) & \text{on } \Sigma_0 \\ \nu \wedge E(\rho,\phi,z) = \displaystyle\sum_{n \in \mathbb{Z}^+} \sum_{m \in \mathbb{Z}^+_*} \left(a_{nm}^{1,TE}\, V_{nm} + a_{nm}^{1,TM}\, W_{nm}\right) & \text{on } \Sigma_1 \\ \nu \wedge E(\rho,\phi,z) = \displaystyle\sum_{n \in \mathbb{Z}^+} \sum_{m \in \mathbb{Z}^+_*} \left(a_{nm}^{2,TE}\, V_{nm} + a_{nm}^{2,TM}\, W_{nm}\right) & \text{on } \Sigma_2 \end{array}\right.$$

Then the problem becomes:

$$
\left\{
\begin{array}{l}
\text{find E} \in V,\ (a_{nm}^{0,TE}, a_{nm}^{0,TM}),\ (a_{nm}^{1,TE}, a_{nm}^{1,TM}) \text{ and } (a_{nm}^{2,TE}, a_{nm}^{2,TM}) \text{ such that },\\
\forall F \in V,\ (b_{nm}^{0,TE}, b_{nm}^{0,TM}),\ (b_{nm}^{1,TE}, b_{nm}^{1,TM}) \text{ and } (b_{nm}^{2,TE}, b_{nm}^{2,TM}) \text{ we have}\\
\displaystyle\int_\Omega Curl\ E \cdot Curl\ \bar{F} d\Omega - k^2 \int_\Omega E \cdot \bar{F} d\Omega\ - \\
\displaystyle\int_{\Sigma_1} R_1(\nu \wedge E) \cdot (\nu \wedge \bar{F}) d\sigma + \int_{\Sigma_2} R_2(\nu \wedge E) \cdot (\nu \wedge \bar{F}) d\sigma\ + \\
\int_{\Sigma_0} R_0(\nu \wedge E - \nu \wedge E^i) \cdot (\nu \wedge \bar{F}) d\sigma = -\displaystyle\int_{\Sigma_0} Curl\ E^i \cdot (\nu \wedge \bar{F}) d\sigma \\
\\
\displaystyle\sum_{l=0}^{2}\sum_{p=0}^{N}\sum_{q=1}^{M}\int_{\Sigma_l} (\nu \wedge E).\left(\bar{b}_{pq}^{l,TE}\ \bar{V}_{pq} + \bar{b}_{pq}^{l,TM}\ \bar{W}_{pq}\right) d\sigma = \\
\displaystyle\sum_{l=0}^{2}\sum_{p=0}^{N}\sum_{q=1}^{M}\sum_{m=1}^{M}\int_{\Sigma_l} \left(a_{nm}^{l,TE}\ V_{nm} + a_{nm}^{l,TM}\ W_{nm}\right)\left(\bar{b}_{pq}^{l,TE}\ \bar{V}_{pq} + \bar{b}_{pq}^{l,TM}\ \bar{W}_{pq}\right) d\sigma + \\
\displaystyle\sum_{p=0}^{N}\sum_{q=1}^{M}\int_{\Sigma_0} (\nu \wedge E^i).\left(\bar{b}_{pq}^{0,TE}\ \bar{V}_{pq} + \bar{b}_{pq}^{0,TM}\ \bar{W}_{pq}\right) d\sigma
\end{array}
\right.
$$

To solve this system, we use numerical approach coupling finite element method and modal expansion method ([5]). In fact, the corresponding matrix has the form:



Where $A_{EF,F}$ represents the interaction between finite element method and the modes. Let X_{EF} be the finite elements degrees of freedom and X_F be the unknown coefficients of the modes. Then we have to solve the system:

$$
\left\{
\begin{array}{rcccccl}
A_{EF,EF} & X_{EF} & + & A_{EF,F} & X_F & = & b \\
A_{F,EF} & X_{EF} & + & A_{F,F} & X_F & = & 0
\end{array}
\right.
$$

Solving this system gives directly the modes. The reflected part an Σ_0 can be calculated using the relation $E^d = E - E^i$.

7. Numerical result

For the numerical validation and to avoid dispersions, we consider the frequency $\frac{109.8=1.25\times 87.843}{a(mm)} < f(GHZ) < \frac{114.754}{a(mm)}$ which excites only the first mode TE_{11}. In this case, the Dirichlet-Neuman operator is ([5]) $R = i\beta^1_{11}$. Denoting by $A = a^{1,TE}_{11}$, we have on a cross section (ν the unitary normal on this section):

$$\left\{\begin{array}{rcl} \nu \wedge E^{TE}_{11} & = & M^{\rho}_{1,1}\, e_\rho + M^{\phi}_{1,1}\, e_\phi = A\left(k^1_{11} J_1^{'}(k^1_{11}\rho) e_\rho + \dfrac{i}{\rho} J_1(k^1_{11}\rho) e_\phi\right) e^{i\phi} e^{i\beta^1_{11} z} \\ & = & A\dfrac{1.841}{2a}\left(J_0(\dfrac{1.841\rho}{a})(e_\rho + ie_\phi) + iJ_2(\dfrac{1.841\rho}{a})(e_\phi + ie_\rho)\right) e^{i\phi} e^{i\beta^1_{11} z}, \end{array}\right.$$

which will be denoted by $\nu \wedge E^{TE}_{11} = A\ \ V^{TE}_{11}$, where $\beta^1_{11} = \sqrt{k^2 - (k^1_{11})^2}$ and $k^1_{11} = 1.841/a$. We consider as junction P a portion in the middle of a circular waveguide of radius $a = 1mm$ and length L (see Figure 3). Theoretically, there is no reflected modes and the electromagnetic wave propagates trough the cable. For the numerical comparison, we take as incident wave the first mode (TE_{11}) with a coefficient $A = 1$ and a frequency $f = 112Ghz$. The mesh contains 10 points per wavelength and the program, written in C++, computes respectively the transmitted and reflected parts $a^{1,TE}_{11}$ and $a^{0,TE}_{11}$ as function of the length of the cable. The next table (Figure 4) shows numerical results.

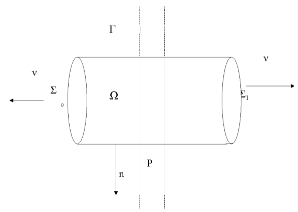


Figure 3: Circular waveguide

As we see, the numerical results gives an acceptable values of the transmitted and reflected parts. The table shows that this numerical approach is independent from the length of the cable. The Figure 5 shows the transmitted and reflected coefficients frequency dependent for $L = 1.mm$. We remark that for $110GHZ < f < 114.754GHZ$ the first mode propagates, but for $87.843GHZ < f(GHZ) < 110GHZ$ we have some dispersion because we are near the cutoff frequency of the first mode.

This numerical example constitutes a validation of the method which can be applied for complicate and industrial applications.

length of the cable	.5 mm	1.mm	1.5 mm	2.mm	3.mm
transmitted part	0.999643	0.99927	0.999655	0.999501	0.999494
reflected part	0.026704	0.038199	0.026274	0.031572	0.031801

Figure 4: Table results

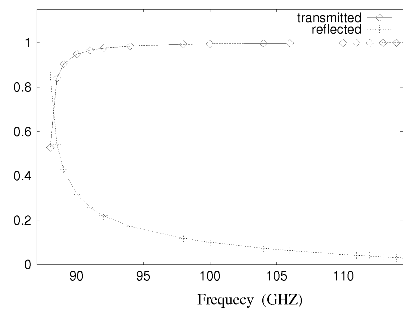


Figure 5: Frequency dependent

8. Conclusion

To study the effect of the junction on the electromagnetic propagation in waveguides, we pass to a bounded problem using the Dirichlet-Neuman operator and we couple the finite element method with the modes. Taking an arbitrary electromagnetic field coming from Σ_0, we calculate the reflected part in Σ_0 and the transmitted parts in Σ_1 and Σ_2.

References

[1] M. D. Deshpande, C. J. Reddy and M. C. Bailey. Analysis of waveguide junction discontinuities and gaps using finite element method, *Electromagnetics*, **18**, No 1, 1998, 81-97.

[2] Fred Gardiol. *Traité d'électricité*, Volume **XIII**, Hyperfréquences, Press Polytechniques et universitaires romandes, École Polytechnique de Lausanne , 1990.

[3] V. N. Kanellopoulos and J. P. Webb. A complete E-plane analysis of waveguide junction using the finite element method, *IEEE Transactions on Microwave Theory and Techniques* **38**, No 3, 1990.

[4] E. Limiti, E. Martini, G. Pelosi, M. Pierozzi and S. Selleri. Efficient hybrid finite elements-modal expansion method for microstrip-to-waveguide transitions analysis, *J. of Electromagn. Waves and Appl.* **15**, No 8, 2001, 1027-1035.

[5] T. Sayah and B. Azar. Coupling finite element method - Modal expansion method for the junction of electromagnetic waveguides, *Intern. J. Appl. Math.* **12**, No 3, 2003, 221-234.

[6] Z. Shen, C. Look and C. Qian. Hybrid finite-element-modal-expansion method for Mmtched magic T-junction, *IEEE Transaction on Magnetics* **38**, No 2, 2002, 385-388.

[7] T. Sieverding and F. Arndt. Modal analysis of the magic tee, *IEEE Microwave Guiwave Lett.* Vol **3**, 1993, 150-152.

[8] J. Taysch and J. Butler. Efficient analysis of periodic dielectric waveguides using Dirichlet-to-Neumann maps, *J. Opt. Soc. Amer. A* **19**, No 6, 2002, 1120-1128.

[9] J. Taysch and J. Butler. Floquet multipliers of periodic waveguides via Dirichlet-to-Neumann maps, *J. Comput. Phys.* **159**, 2000, 90-102.

[1] *Département de Mathématiques*
Faculté des Sciences
Université Saint Joseph
BP 11-514, Riad El Solh
Beirut 1107 2050, LEBANON
email: tsayah@fs.usj.edu.lb

Received 30.09.2003

[2] *L.P.S.E. Faculté des sciences II-Fanar*
Université Libanaise
BP 90656, Jdeidet, LEBANON
email: bazar@ul.edu.lb