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On Some Special General Integrals of a Linear Differential Equation of Second Order

Ilija A. Shapkarev[1], *Boro M. Piperevski*[2], *Elena J. Hadzieva*[3]

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A linear homogeneous differential equation of second order with polynomial coefficients is considered in this article. The degree of each polynomial is equal to the order of the derivative of the unknown function which is multiplied by. By the methods of differentiating and integrating, we obtain the conditions for existence of the two polynomial solutions of the differential equation as well its general solution. These results are applied to three other differential equations that have the same normal kind as the original equation, so three other special general integrals of the original equation are obtained.

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1.

Consider the homogeneous linear differential equation of second order

$$\alpha y'' + \beta y' + Fy = 0, \tag{1.1}$$

where $\alpha(x) = Ax^2 + \beta x + C$, $\beta(x) = Dx + E$; $A \neq 0, B, C, D, E$ are constants. Provided that m is a positive integer, we differentiate (1.1) m-times and get the differential equation of $(m+2)$-nd order, see [3,4]:

$$\alpha y^{(m+2)} + (m\alpha' + \beta)y^{(m+1)} + \left(\binom{m}{2}\alpha'' + \binom{m}{1}\beta' + F\right)y^{(m)} = 0. \tag{1.2}$$

As it is known ([1,3]), the differential equation (1.1) has a polynomial solution of degree m if and only if its characteristic equation

$$\binom{\xi}{2}\alpha'' + \binom{\xi}{1}\beta' + F = 0 \tag{1.3}$$

has at least one root $\xi^* = m$. If the other root is also a positive integer, then m is the smaller one.

In order to find a polynomial solution of the differential equation (1.1) as well its general solution, we use the substitution

$$y = \alpha e^{-\int \frac{\beta}{\alpha} dx} w \tag{1.4}$$

which transforms (1.1) in the following differential equation

$$\alpha w'' + (2\alpha' - \beta)w' + (\alpha'' - \beta' + F)w = 0.$$

Integrating the above equation m-times leads to the equation

$$\begin{aligned} \alpha w^{(-m+2)} &- \left[(m-2)\alpha' + \beta\right] w^{(-m+1)} \\ &+ \left[\binom{m-1}{2}\alpha'' + \binom{m-1}{1}\beta' + F\right] w^{(-m)} = 0, \end{aligned}$$

where $w^{(-m)} = \int\int \ldots \int w dx^m$. If we integrate this equation once more, we get the equation

$$\alpha w^{(-m+1)} - \left[(m-1)\alpha' + \beta\right] w^{(-m)} = A_2,$$

where A_2 denotes an arbitrary constant. Solving the last equation, for $w^{(-m)}$ we get

$$w^{(-m)} = A_1 \left(\alpha^{m-1} e^{\int \frac{\beta}{\alpha} dx}\right)^{(m)} + A_2 \left(\alpha^{m-1} e^{\int \frac{\beta}{\alpha} dx} \int \alpha^{-m} e^{-\int \frac{\beta}{\alpha} dx} dx\right)^{(m)},$$

where A_1 is an integration constant.

According to the substitution (1.4), the general solution of differential equation (1.1) is

$$\begin{aligned} y &= A_1 \alpha e^{-\int \frac{\beta}{\alpha} dx} \left(\alpha^{m-1} e^{\int \frac{\beta}{\alpha} dx}\right)^{(m)} \\ &+ A_2 \alpha e^{-\int \frac{\beta}{\alpha} dx} \left(\alpha^{m-1} e^{\int \frac{\beta}{\alpha} dx} \int \alpha^{-m} e^{-\int \frac{\beta}{\alpha} dx} dx\right)^{(m)}. \end{aligned}$$

Now, let us suppose that the original equation has another polynomial solution of degree $(m+n)$, where n is a natural number. Applying (1.3) to (1.2), for $y^{(m+1)}$ we get

$$y^{(m+1)} = \alpha^{-m} e^{-\int \frac{\beta}{\alpha} dx},$$

i.e.

$$y = \left(\alpha^{-m} e^{-\int \frac{\beta}{\alpha} dx}\right)^{(-m-1)}.$$

Hence, the differential equation (1.1) would have a second polynomial solution of $(m+n)$-th degree if and only if, see [2,5]:

$$\left(\alpha^{-m}e^{-\int \frac{\beta}{\alpha}dx}\right)^{(n)} = 0,$$

which means

$$\alpha^{-m}e^{-\int \frac{\beta}{\alpha}dx} = P_{n-1}(x).$$

From the above:

$$\beta(x) = -m\alpha' - \alpha\frac{P'_{n-1}(x)}{P_{n-1}(x)}.$$

If x_1 and x_2 are the zeros of the polynomial $\alpha(x)$, for P_{n-1} we get:

$$P_{n-1}(x) = (x-x_1)^k(x-x_2)^{n-1-k} \qquad k = 0,1,\dots,n-1,$$

and for $\beta(x)$,

$$\beta(x) = -A(m+k)(x-x_2) - A(m+n-1-k)(x-x_1).$$

Furthermore,

$$\begin{aligned} D &= -(2m+n-1)A, \\ F &= -\binom{m}{2}\alpha'' - \binom{m}{1}\beta' = m(m+n)A. \end{aligned}$$

Finally, we can say that the general solution of the differential equation

$$\alpha y'' - \left(m\alpha' - \alpha\frac{P'_{n-1}(x)}{P_{n-1}(x)}\right)y' + m(m+n)Ay = 0,$$

i.e.

$$\alpha y'' - \left[(m+k)(x-x_2) + (m+n-1-k)(x-x_1)\right]Ay' + m(m+n)Ay = 0$$

is a polynomial of $(m+n)$-th degree

$$\begin{aligned} y &= A_1\alpha^{m+1}P_{n-1}(x)\left(\frac{1}{\alpha P_{n-1}(x)}\right)^{(m)} \\ &+ A_2\alpha^{m+1}P_{n-1}(x)\left[\frac{1}{\alpha P_{n-1}(x)}\int P_{n-1}(x)dx\right]^{(m)}, \end{aligned}$$

i.e.

$$\begin{aligned} y &= A_1(x-x_1)^{m+1+k}(x-x_2)^{m+n-k}\left[(x-x_1)^{-k-1}(x-x_2)^{-n+k}\right]^{(m)} \\ &+ A_1(x-x_1)^{m+1+k}(x-x_2)^{m+n-k} \\ &\times \left[(x-x_1)^{-k-1}(x-x_2)^{-n+k}\int (x-x_1)^{k}(x-x_2)^{n-1-k}dx\right]^{(m)}. \end{aligned}$$

In order to obtain other special general integrals of differential equation (1.1) we shall consider three other linear differential equations of second order that have the same normal kind [4] as the differential equation (1.1). We shall apply the above results to these three equations in the case when at least one root of their characteristic equations is a natural number. Using the relations between these three equations and the equation (1.1) we obtain new special general integrals as follows.

2.

The differential equation

$$\alpha z'' + \beta_1 z' + F_1 z = 0, \tag{2.1}$$

where $\beta_1 = 2\alpha' - \beta$, $F_1 = \alpha'' - \beta' + F$, has a characteristic equation

$$\binom{\eta}{2}\alpha'' + \binom{\eta}{1}\beta_1' + \binom{\eta}{0}F_1 = 0. \tag{2.2}$$

Let us denote its roots by η^* and η^{**} and suppose that at least one of them is a natural number. In the case when they are both natural, η^* will be the smaller one. So, if $\eta^* = m$, m is natural, the general solution of the differential equation (2.1) will be:

$$\begin{aligned} z &= B_1\alpha e^{-\int\frac{\beta_1}{\alpha}dx}\left(\alpha^{m-1}e^{\int\frac{\beta_1}{\alpha}dx}\right)^{(m)} \\ &+ B_2\alpha e^{-\int\frac{\beta_1}{\alpha}dx}\left(\alpha^{m-1}e^{\int\frac{\beta_1}{\alpha}dx}\int\alpha^{-m}e^{-\int\frac{\beta_1}{\alpha}dx}dx\right)^{(m)} \\ &= B_1\alpha^{-1}e^{\int\frac{\beta}{\alpha}dx}\left(\alpha^{m+1}e^{-\int\frac{\beta}{\alpha}dx}\right)^{(m)} \\ &+ B_2\alpha^{-1}e^{\int\frac{\beta}{\alpha}dx}\left(\alpha^{m+1}e^{-\int\frac{\beta}{\alpha}dx}\int\alpha^{-m-2}e^{\int\frac{\beta}{\alpha}dx}dx\right)^{(m)}, \end{aligned}$$

where B_1, B_2 are arbitrary constants.

Since the differential equation (1.1) with the substitution

$$y = \alpha e^{-\int \frac{\beta}{\alpha} dx} z$$

transforms into the differential equation (2.1), its general integral is going to be:

$$y = B_1 \left(\alpha^{m+1} e^{-\int \frac{\beta}{\alpha} dx} \right)^{(m)} + B_2 \left(\alpha^{m+1} e^{-\int \frac{\beta}{\alpha} dx} \int \alpha^{-m-2} e^{\int \frac{\beta}{\alpha} dx} dx \right)^{(m)} .$$

3.

The differential equation

$$\alpha u'' + \beta_2 u' + F_2 u = 0, \tag{3.1}$$

where

$$\begin{aligned} \beta_1 &= \left(2A + \frac{S}{R} \right) x + \frac{B+R}{2A} \left(2A - D + \frac{S}{R} \right) + E, \\ F_2 &= \frac{1}{4A} \left(2A - D + \frac{S}{R} \right) \left(D + \frac{S}{R} \right) + F, \\ S &= BD - 2AE, \quad R = \sqrt{B^2 - 4AC}, \quad R \neq 0, \end{aligned}$$

has a characteristic equation

$$\binom{\zeta}{2} \alpha'' + \binom{\zeta}{1} \beta_2' + \binom{\zeta}{0} F_2 = 0. \tag{3.2}$$

Suppose that the characteristic equation (3.2) has a root $\zeta^* = m$, m is a natural number. If it has another root ζ^{**}, also a natural number, m is the smaller one. The general solution of the differential equation will be

$$\begin{aligned} u &= C_1 \alpha e^{-\int \frac{\beta_2}{\alpha} dx} \left(\alpha^{m-1} e^{\int \frac{\beta_2}{\alpha} dx} \right)^{(m)} \\ &+ C_2 \alpha e^{-\int \frac{\beta_2}{\alpha} dx} \left(\alpha^{m-1} e^{\int \frac{\beta_2}{\alpha} dx} \int \alpha^{-m} e^{-\int \frac{\beta_2}{\alpha} dx} dx \right)^{(m)} . \end{aligned}$$

Since the differential equation (1.1), with the substitution

$$y = \alpha^{\frac{1}{2} + \frac{S}{4AR}} e^{\left(\frac{1}{2} - \frac{D}{4A}\right) R \int \frac{dx}{\alpha} - \frac{1}{2} \int \frac{\beta}{\alpha} dx} u$$

transforms into form (3.1), the general solution of the differential equation (1.1) in this case, is:

$$\begin{aligned} y &= C_1\alpha^{\frac{3}{2}+\frac{S}{4AR}}\, e^{(\frac{1}{2}-\frac{D}{4A})R\int\frac{dx}{\alpha}-\frac{1}{2}\int\frac{\beta}{\alpha}dx-\int\frac{\beta_2}{\alpha}dx}\left(\alpha^{m-1}e^{\int\frac{\beta_2}{\alpha}dx}\right)^{(m)} \\ &+ C_2\alpha^{\frac{3}{2}+\frac{S}{4AR}}\, e^{(\frac{1}{2}-\frac{D}{4A})R\int\frac{dx}{\alpha}-\frac{1}{2}\int\frac{\beta}{\alpha}dx-\int\frac{\beta_2}{\alpha}dx}\times \\ &\times \left(\alpha^{m-1}e^{\int\frac{\beta_2}{\alpha}dx}\int\alpha^{-m}e^{-\int\frac{\beta_2}{\alpha}dx}dx\right)^{(m)}. \end{aligned}$$

4.

The differential equation

$$\alpha v'' + \beta_3 v' + F_3 v = 0, \tag{4.1}$$

where

$$\begin{aligned} \beta_3 &= \left(2A-\frac{S}{R}\right)x + \frac{B-R}{2A}\left(2A-D-\frac{S}{R}\right)+E, \\ F_3 &= \frac{1}{4A}\left(2A-D-\frac{S}{R}\right)\left(D-\frac{S}{R}\right)+F, \end{aligned}$$

has a characteristic equation

$$\binom{\tau}{2}\alpha'' + \binom{\tau}{1}\beta_3' + \binom{\tau}{0}F_3 = 0. \tag{4.2}$$

Suppose that the last equation has a root $\tau^* = m$, m is a natural number. If there exists another root τ^{**}, which is also a natural, m is the smaller one. In that case, the general solution of the differential equation (4.1) is

$$\begin{aligned} v &= D_1\alpha e^{-\int\frac{\beta_3}{\alpha}dx}\left(\alpha^{m-1}e^{\int\frac{\beta_3}{\alpha}dx}\right)^{(m)} \\ &+ D_2\alpha e^{-\int\frac{\beta_3}{\alpha}dx}\left(\alpha^{m-1}e^{\int\frac{\beta_3}{\alpha}dx}\int\alpha^{-m}e^{-\int\frac{\beta_3}{\alpha}dx}dx\right)^{(m)}. \end{aligned}$$

As the differential equation (1.1) with the substitution

$$y = \alpha^{\frac{1}{2}-\frac{S}{4AR}}\, e^{(\frac{D}{4A}-\frac{1}{2})R\int\frac{dx}{\alpha}-\frac{1}{2}\int\frac{\beta}{\alpha}dx}\, v$$

transforms into equation (4.1), its general integral will be:

$$\begin{aligned} y &= D_1\alpha^{\frac{3}{2}-\frac{S}{4AR}}\, e^{(\frac{D}{4A}-\frac{1}{2})R\int\frac{dx}{\alpha}-\frac{1}{2}\int\frac{\beta}{\alpha}dx-\int\frac{\beta_3}{\alpha}dx}\left(\alpha^{m-1}e^{\int\frac{\beta_3}{\alpha}dx}\right)^{(m)} \\ &+ D_2\alpha^{\frac{3}{2}-\frac{S}{4AR}}\, e^{(\frac{D}{4A}-\frac{1}{2})R\int\frac{dx}{\alpha}-\frac{1}{2}\int\frac{\beta}{\alpha}dx-\int\frac{\beta_3}{\alpha}dx} \\ &\times \left(\alpha^{m-1}e^{\int\frac{\beta_3}{\alpha}dx}\int\alpha^{-m}e^{-\int\frac{\beta_3}{\alpha}dx}dx\right)^{(m)}. \end{aligned}$$

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[1,2,3] *Faculty of Electrical Engineering*
P.O. Box 574
1000 Skopje, MACEDONIA

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e-mails: [1,3] *hadzieva@etf.ukim.edu.mk* , [2] *borom@etf.ukim.edu.mk*