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An Inverse Problem with Two Unknown Terms

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In this paper, we consider the problem of determining of two unknown flux terms on the boundaries of parabolic type problem. The unknown terms are determined from two unknown overspecified conditions.

The uniqueness of solutions of the inverse problem and a system of non-linear Volterra integral equations will be shown.

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1. Introduction

The purpose of this paper is to identify two unknown functions in a linear heat conduction problem from some overspecified data measured on the boundary. In pursuit of this, for $T > 0$, let $Q_T = \{ (x,t) \mid 0 < x < 1, 0 < t < T\}$ and consider the problem of determining the pair of functions $(u, (\phi, \psi))$ which satisfy

$$u_t - u_{xx} = \gamma(x,t), \qquad 0 < x < 1,\ 0 < t < T, \tag{1}$$

$$u(x,0) = f(x), \qquad 0 < x < 1, \tag{2}$$

$$u_x(0,t) = \phi(t, u(0,t)), \qquad 0 < t < T, \tag{3}$$

$$-u_x(1,t) = \psi(t, u(1,t)), \qquad 0 < t < T, \tag{4}$$

and overspecified conditions

$$u(0,t) = g(t), \quad 0 < t < T, \tag{5}$$

and

$$u(1,t) = h(t), \quad 0 < t < T. \tag{6}$$

It is clear that for any given ϕ and ψ there may be no solution $u(x,t)$ satisfying all of the conditions (1)-(6). On the other hand, when ϕ is known, a priori, then under certain conditions there may exist a unique solution $u(x,t)$ of the problem (1)-(4). If for some choices of ϕ and ψ, the solution to (1) through (4) satisfies (5) too, then we say the pair of functions $(u,(\phi,\psi))$ provides a solution to the inverse problem(1)-(5).

The problem of determining unknown flux on the boundaries in parabolic differential equations has been treated previously by many authors [1-6].

A physical example of such problem arises from the one-dimensional conduction of heat in a homogeneous bar of length 1, with one end located at the origin. Say, let the initial temperature and the source term be given, and it be heated at $x=0$ and $x=1$, with conditions determined at any time.

2. Existence and uniqueness results

In order to solve the inverse problem (1)-(6), let us consider the following direct problem

$$u_t - u_{xx} = \gamma(x,t), \qquad 0<x<1,\ 0<t<T, \tag{7}$$

$$u(x,0) = f(x), \qquad 0<x<1, \tag{8}$$

$$u(0,t) = g(t), \qquad 0<t<T, \tag{9}$$

$$u(1,t) = h(t), \qquad 0<t<T. \tag{10}$$

Theorem 1. *Let us assume that the source function $\gamma(x,t)$ is bounded over its domain and uniformly Hölder continuous on each compact subset of the domain under consideration, and also assume that the initial boundary data are piecewise continuous. Then the problem (7) – (10) possesses a unique bounded solution in the form*

$$\begin{aligned} u(x,t) &= \int_0^1 \{\theta(x-\xi,t)-\theta(x+\xi,t)\}\ f(\xi)\ d\xi \\ &- 2\int_0^t \frac{\partial\theta(x,t-\tau)}{\partial x}\ g(\tau)\ d\tau + 2\int_0^t \frac{\partial\theta(x-1,t-\tau)}{\partial x}\ h(\tau)\ d\tau \\ &+ \int_0^1\int_0^t \{\theta(x-\xi,t-\tau)-\theta(x+\xi,t-\tau)\}\ \gamma(\xi,\tau)\ d\tau\ d\xi, \end{aligned} \tag{11}$$

where

$$\theta(x,t) = \sum_{m=-\infty}^{\infty} k(x+2m,t)$$

with

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}), \qquad t > 0.$$

Here θ and k are called θ-function and fundamental solution of heat equation, respectively [1].

Differentiating with respect to x, we obtain

$$\begin{aligned} u_x(x,t) &= \int_0^1 \{\frac{\partial\theta}{\partial x}(x-\xi,t) - \frac{\partial\theta}{\partial x}(x+\xi,t)\}\ f(\xi)\ d\xi \\ &- 2\int_0^t \frac{\partial^2\theta}{\partial x^2}(x,t-\tau)\ g(\tau)\ d\tau + 2\int_0^t \frac{\partial^2\theta}{\partial x^2}(x-1,t-\tau)\ h(\tau)\ d\tau \\ &+ \int_0^1\int_0^t \{\frac{\partial\theta}{\partial x}(x-\xi,t-\tau) - \frac{\partial\theta}{\partial x}(x+\xi,t-\tau)\}\ \gamma(\xi,\tau)\ d\tau\ d\xi. \end{aligned}$$

By the following θ-function properties

$$\begin{aligned} &(a) & \frac{\partial\theta}{\partial x}(x-\xi,t) &= -\frac{\partial\theta}{\partial\xi}(x-\xi,t), \\ &(b) & \frac{\partial\theta}{\partial x}(x+\xi,t) &= \frac{\partial\theta}{\partial\xi}(x+\xi,t), \\ &(c) & \frac{\partial^2\theta}{\partial x^2}(x,t-\tau) &= \frac{\partial\theta}{\partial t}(x,t-\tau) = -\frac{\partial\theta}{\partial\tau}(x,t-\tau), \\ &(d) & \lim_{\tau\to t}\theta(x,t-\tau) &= 0,\ \ 0<x<1, \\ &(e) & \theta(x-1,t) &= \theta(x+1,t), \\ &(f) & \theta(-\xi,t) &= \theta(\xi,t), \\ &(g) & \theta(1-\xi,t) &= \theta(1+\xi,t), \end{aligned}$$

and integrating by parts yields

$$\begin{aligned} u_x(x,t) &= \int_0^1 \{\theta(x-\xi,t) + \theta(x+\xi,t)\}\ f'(\xi)\ d\xi \\ &- 2\int_0^t \theta(x,t-\tau)\ g'(\tau)\ d\tau + 2\int_0^t \theta(x-1,t-\tau)\ h'(\tau)\ d\tau \\ &+ \int_0^1\int_0^t \{\theta(x-\xi,t-\tau) + \theta(x+\xi,t-\tau)\}\ \gamma_\xi(\xi,\tau)\ d\tau\ d\xi. \end{aligned} \tag{12}$$

Now, from (f),(g),(3), and (12) we find:

$$\phi(t,g(t)) = 2\int_0^1 \theta(\xi,t)\ f'(\xi)\ d\xi$$

$$- \quad 2\int_0^t \theta(0,t-\tau)\ g'(\tau)\ d\tau + 2\int_0^t \theta(-1,t-\tau)\ h'(\tau)\ d\tau$$
$$+ \quad 2\int_0^1\int_0^t \theta(\xi,t)\}\ \gamma_\xi(\xi,\tau)\ d\tau\ d\xi.$$

If we assume that the function $s = g(t)$ is invertible, then we have

$$\phi(s) = 2\int_0^1 \theta(\xi, g^{-1}(s))\ f'(\xi)\ d\xi$$
$$- \quad 2\int_0^{g^{-1}(s)} \theta(0, g^{-1}(s)-\tau)g'(\tau)\ d\tau + 2\int_0^{g^{-1}(s)} \theta(-1, g^{-1}(s)-\tau)\ h'(\tau)\ d\tau$$
$$+2\int_0^1\int_0^{g^{-1}(s)} \theta(\xi, g^{-1}(s))\ \gamma_\xi(\xi,\tau)\ d\tau\ d\xi. \tag{13}$$

A similar result may be obtained for $\psi(s)$. In fact, by employing (4),(11),$u(1,t) = h(t)$, and assuming $v = h(t)$ to be invertible, we obtain

$$\psi(v) = -2\int_0^1 \theta(\xi+1, h^{-1}(v))\ f'(\xi)\ d\xi$$
$$+2\int_0^{h^{-1}(v)} \theta(1, h^{-1}(v)-\tau)\ g'(\tau)\ d\tau - 2\int_0^{h^{-1}(v)} \theta(0, h^{-1}(v)-\tau)\ h'(\tau)\ d\tau$$
$$-2\int_0^1\int_0^{h^{-1}(v)} \theta(1+\xi, h^{-1}(v))\ \gamma_\xi(\xi,\tau)\ d\tau\ d\xi. \tag{14}$$

Using Theorem 1, it is easy to show that the pair of solution $(u,(\phi,\psi))$, the solution of the inverse problem (1)-(6), exists and is unique. For any given Lipschitz continuous f, the problem (1)-(4) has a unique solution in the form of [1]:

$$u(x,t) = \int_0^1 \{\theta(x-\xi,t) + \theta(x+\xi,t)\}\ f(\xi)\ d\xi$$
$$-2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)\phi(\tau,\phi_1(\tau))d\tau + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)\psi(\tau,\phi_2(\tau))d\tau$$
$$+\int_0^1\int_0^t \{\theta(x-\xi,t-\tau) + \theta(x+\xi,t-\tau)\}\ \gamma(\xi,\tau)\ d\tau\ d\xi, \tag{15}$$

if and only if ϕ_1, ϕ_2 are piecewise continuous solution of the following system of non-linear Volterra integral equations

$$\phi_1(t) = w(0,t) - 2\int_0^t \theta(0,t-\tau)\ \phi(\tau,\phi_1(\tau))\ d\tau + 2\int_0^t \theta(-1,t-\tau)\ \psi(\tau,\phi_2(\tau))\ d\tau$$

$$+2\int_0^1\int_0^t \theta(\xi, t-\tau)\ \gamma(\xi,\tau)\ d\tau\ d\xi, \tag{16}$$

$$\phi_2(t) = w(1,t) - 2\int_0^t \theta(1,t-\tau)\ \phi(\tau,\phi_1(\tau))\ d\tau + 2\int_0^t \theta(0,t-\tau)\ \psi(\tau,\phi_2(\tau))\ d\tau$$

$$+2\int_0^1\int_0^t \theta(\xi+1, t-\tau)\ \gamma(\xi,\tau)\ d\tau\ d\xi, \tag{17}$$

where

$$w(x,t) = \int_0^1 \{\theta(x-\xi,t) + \theta(x+\xi,t)\}\ f(\xi)\ d\xi. \tag{18}$$

We summarize the above discussion by the following statement:

Theorem 2. *For any given Lipschitz continuous functions ϕ and ψ, and piecewise continuous functions f,g, and h in problem (1)-(5), the pair of solutions (ϕ_1, ϕ_2) for the system of non-linear Volterra integral equations (15) and (16) exists and is unique.*

Moreover, the solution for ϕ, ψ may be obtained from (12) and (13), respectively.

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