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Function Specification Method for Determination an Unknown Term in a Nonlinear Diffusion Equation

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In this paper, we consider a nonlinear parabolic problem. We use a whole domain estimation algorithm for identifying an unknown term from an overspecified condition. The temperature history is not specified at $x = 0$, but instead a measured temperature history is given on the interior point. Another aim of this paper is to find the conditions which this method becomes stable.

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1. Introduction

Quantitative studies of the heat conduction problems processes occurring in the industrial application, require accurate knowledge of the surface conditions. Such problems arise in a variety of physical situations. For example, in radiative heat transfer. Suich type problem will be considered in this paper. The problem involved in this paper has been discussed by many authors, especially, Cannon and Zachmann [5], Cannon and Duchateau [4], Shidfar [7]. The inverse heat conduction problems (IHCP) are extremely ill-posed in sense of continuous dependence upon the data [8]. That is why special regularization method are needed to solve these type problems.

Consider the one-dimensional parabolic inverse problem

$$u_t = u_{xx}, \qquad 0 < x < 1,\ 0 < t < T, \tag{1}$$
$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{2}$$
$$u(1,t) = g(t), \qquad 0 \le t \le T, \tag{3}$$
$$u_x(0,t) = P((u(0,t)), \qquad 0 \le t \le T, \tag{4}$$

where $f(x)$, $g(t)$ are given sufficiently smooth functions in their domains, and $u(x,t)$ and $P(u(0,t))$ are unknown functions which remain to be determined. To solve this inverse problem, we use the overspecified condition

$$u(x_1,t) = Y(t), \qquad 0 < t \le T, \tag{5}$$

where x_1 is a given fixed point in the $(0,1]$. In practice, temperature at x_1 are measured sequentially at time interval Δt, from Δt to $n\Delta t$, where n is a given natural number. Designate these temperature by $Y_i \quad i = 1,2,\ldots,n$, equation (5) is replaced by

$$u(x_1, i\Delta t) = Y_i\ , \quad i = 1,2,\ldots,n. \tag{6}$$

As a consequence, the solution for heat flux at $x = 0$, becomes

$$-\left.\frac{\partial u(x,i\Delta t)}{\partial x}\right|_{x=0} = q(i\Delta t), \quad i = 1,2,\ldots,n. \tag{7}$$

The goal of a solution method for inverse heat conduction problem (IHCP) is to obtain $q(t)$ that is a best approximation to $u_x(0,t)$ and not sensitive to temperature measurement errors. The problem (1)-(5) may be divided into two direct and inverse problems.

2. Direct and inverse problems

For piecewise-continuous functions f, Y and g, we consider the following direct problem

$$u_t = u_{xx}\ , \qquad x_1 < x < 1,\ 0 < t < T, \tag{8}$$
$$u(x,0) = f(x)\ , \qquad x_1 \le x \le 1, \tag{9}$$
$$u(x_1,t) = Y(t)\ , \qquad 0 \le t \le T, \tag{10}$$
$$u(1,t) = g(t)\ , \qquad 0 \le t \le T. \tag{11}$$

By a translation and a relabeling of variables, the solution to the above problem is (see [3]):

$$\begin{aligned} u(x,t) &= \int_0^1 \{\theta(x-\xi,t)-\theta(x+\xi,t)\}f(\xi)d\xi - 2\int_0^t \frac{\partial\theta}{\partial x}(x-x_1,t-\tau)Y(\tau)d\tau \\ (12) \qquad &+ 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)g(\tau)d\tau, \end{aligned}$$

where

$$(13.a) \qquad \theta(x,t) = \sum_{m=-\infty}^{+\infty} k(x+2m,t), \quad t>0,$$

and

$$(13.b) \qquad k(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\{-\frac{x^2}{4t}\}.$$

The heat flux at x_1 may be found from the solution for temperature distribution in $x_1 \le x \le 1$, by using

$$(14) \qquad \left.\frac{\partial u}{\partial x}\right|_{x=x_1} = h(t).$$

The problem

$$\begin{array}{llll} (15) & u_t = u_{xx}\,, & 0<x<x_1,\ 0<t<T, \\ (16) & u(x,0) = f(x)\,, & 0\le x\le x_1, \\ (17) & u_x(x_1,t) = h(t)\,, & 0\le t\le T, \\ (18) & u_x(0,t) = q(t)\,, & 0\le t\le T, \end{array}$$

where $u(x,t)$ and $u_x(0,t) = q(t)$ are required over a finite time interval, $0 \le t \le T$, is an inverse problem.

This inverse problem will be considered in the next section.

3. Inverse heat conduction estimation procedure

In order to solve the inverse problem (15)-(18), let us use the superposition principle

$$(19) \qquad u(x,t) = v(x,t) + w(x,t),$$

where $v(x,t)$ is the solution of the problem:

$$\begin{aligned}
&(20) \qquad & v_t &= v_{xx}\ , & & 0 < x < x_1,\ 0 < t < T,\\
&(21) \qquad & v(x,0) &= f(x)\ , & & 0 \le x \le x_1,\\
&(22) \qquad & v_x(x_1,t) &= h(t)\ , & & 0 \le t \le T,\\
&(23) \qquad & v_x(0,t) &= 0\ , & & 0 \le t \le T,
\end{aligned}$$

and $w(x,t)$ satisfying

$$\begin{aligned}
&(24) \qquad & w_t &= w_{xx}\ , & & 0 < x < x_1,\ 0 < t < T,\\
&(25) \qquad & w(x,0) &= 0\ , & & 0 \le x \le x_1,\\
&(26) \qquad & w_x(x_1,t) &= 0\ , & & 0 \le t \le T,\\
&(27) \qquad & w_x(0,t) &= q(t)\ , & & 0 \le t \le T.
\end{aligned}$$

For the piecewise continuous functions $f(x)$ and $h(t)$ the problem (20)-(23) has the solution ([3]):

$$(28) \qquad v(x,t) = \sum_{n=0}^{\infty} b_n \exp\{-n^2\pi^2 t\} \cos n\pi x + 2\int_0^1 \theta(x-x_1, t-\tau)h(\tau)d\tau,$$

where

$$\begin{aligned}
&(29.a) \qquad & b_n &= 2\int_0^1 f(x)\cos n\pi x dx,\\
&(29.b) \qquad & b_0 &= \int_0^1 f(x)dx.
\end{aligned}$$

Now, for $u_x(0,t) = q(t)$, let us to consider the following form

$$(30) \qquad q(t) = \sum_{j=0}^{r-1} \beta_j t^j\ , \qquad 0 \le t \le T.$$

By this assumption, one may use whole domain estimation algorithm [1] to solve the problem (24)-(27). By a direct substitution

$$(31) \qquad w(x,t) = \sum_{j=0}^{r-1} \beta_j \phi^{(j)}(x,t),$$

into the problem (24)-(27), where $\phi^{(j)}(x,t),\ j = 0,1,2,\dots,r-1$, are solutions of following problems

$$\begin{aligned}
&(32) \qquad & \phi_t^{(j)} &= \phi_{xx}^{(j)}\,, & & 0 < x < x_1,\ 0 < t < T,\\
&(33) \qquad & \phi^{(j)}(x,0) &= 0\,, & & 0 \le x \le x_1,\\
&(34) \qquad & \phi_x^{(j)}(x_1,t) &= 0\,, & & 0 \le t \le T,\\
&(35) \qquad & \phi_x^{(j)}(0,t) &= t^j\,, & & 0 \le t \le T, \quad j = 0,1,2,\dots,r-1.
\end{aligned}$$

We will obtain a best approximation to the solution for the problem (24)-(27). For this purpose let us suppose that temperatures at times $t_i \in [0,T],\quad i = 1,2,\dots,n$, are measured at $x = x_1$ and are denoted by $Y_1, Y_2, \dots, Y_n$, respectively. Then we apply the least squares method to minimize

$$S = \sum_{i=1}^{n} (Y_i - \sum_{j=0}^{r-1} \beta_j \phi_i^{(j)})^2, \tag{36}$$

with respect to $\beta_0, \beta_1, \dots, \beta_{r-1}$. Then we take the first derivatives of S with respect to $\beta_0, \beta_1, \dots, \beta_{r-1}$, and set the r obtained equations equal to zero. Now, if $\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_{r-1}$, are the solutions for the coefficients in (36). In fact, we find the set of linear equations

$$\sum_{i=1}^{n} [Y_i - \sum_{j=0}^{r-1} \hat{\beta}_i \phi_i^{(j)}](-\phi_i^{(m)}) = 0\,, \quad m = 0,1,\dots,r-1, \tag{37}$$

which is of the form $\mathbf{A}\beta = \mathbf{b}$, where $\beta = (\beta_0, \beta_1, \dots, \beta_{r-1})$, and

$$b_s = \sum_{i=1}^{n} Y_i \phi_i^{(s)}\,, \quad s = 0,1,\dots,r-1, \tag{38}$$

and $\mathbf{A}$ is a symmetric matrix with entries

$$a_{ls} = \sum_{i=1}^{n} \phi_i^{(\ell)} \phi_i^{(s)}\,, \quad \ell, s = 0,1,\dots,r-1\,, \ell \le\ s. \tag{39}$$

Then (31) with these coefficients will be a best approximation for $q(t)$. Now, putting $u(0,t_i)$ and $u_x(0,t_i)$ for $i = 1,2,\dots,n$, in (4)

$$-u_x(0,t_i) + P(u(0,t_i)) = 0\,, \quad i = 1,2,\dots,n, \tag{40}$$

and using the interpolation method, we find a polynomial interpolation of degree $\le n$ for P. Now, the above results may summarize in the following statement:

Theorem 1. *For any piecewise continuous functions $f(x)$, $g(x)$ and $Y(t)$, the inverse problem (1)-(5) has a unique solution pair $\Big(u(x,t), P(u(0,t))\Big)$, where $u(x,t)$ has a representation in the form $u(x,t) = v(x,t) + w(x,t)$, and P is a polynomial of degree $\leq n$.*

4. Regularization

The whole domain estimation algorithm introduced by Beck, is considered as one of the most efficient methods for the inverse heat conduction problems which are extremely ill-posed and time dependent, [1]. When one solves the inverse problem (15)-(18) an estimated value of $\mathbf{q} = [q(t_1), q(t_2), \ldots, q(t_n)]$, will be produced. The goal is to make estimated $u(x_1, q(t_i))$, as close as possible to the exprimental data $Y(t_j)$, for $j = 1, 2, \ldots, n$. The whole domain regularization of IHCP, find the boundary heat flux $q(t)$ at on the $x = 0$, that minimizes the object functional

$$S(q) = \sum_{j=1}^{n} \Big(Y(t_j) - u(x_1, q(t_j)) \Big)^2 + \alpha \sum_{j=1}^{n} q_j^2, \tag{41}$$

where α is the regularization parameter. This is called a whole domain zeroth-order regularization procedure. In order to help the stabilization of the inverse problem, it is assumed that the unknown function $q(t)$ itself satisfies a known prescribed L_2 bound, $\|q\|_2 \leq E$. Using the finite integral transform technique [9], the problem (24)-(27) becomes equivalent with a Volterra integral equation of the first kind for the unknown function $q(t)$:

$$w(x,t) = \frac{1}{x_1} \int_{s=0}^{t} q(s)ds + \frac{2}{x_1} \sum_{m=1}^{\infty} \cos(\frac{m\pi}{x_1} x) \int_0^t \exp\{-\frac{m^2\pi^2}{x_1^2}(t-s)\} q(s)ds. \tag{42}$$

Actually, at $x = x_1$ we have

$$w(x_1,t) = \frac{1}{x_1} \int_{s=0}^{t} q(s)ds + \frac{1}{x_1} \int_{s=0}^{t} q(s)k(t,s)ds\ , \quad t \geq 0, \tag{43}$$

where the convolution kernel $k(t,s)$ is given by

$$k(t,s) = \sum_{m=1}^{\infty} 2(-1)^m \exp\{-\frac{m^2\pi^2}{x_1^2}(t-s)\}. \tag{44}$$

From (6), (28), and (19) we have

$$w_j = u(x_1,t_j) - v(x_1,t_j)\ ; \quad j = 1, 2, \ldots, n, \tag{45}$$

suppose that $q(t)$ can be expressed as [6]

$$q(t) = \sum_{k=0}^{r-1} \beta_k T_k(t), \tag{46}$$

where $T_k(t)$ is the k^{th} Chebyshev polynomial of the first kind. The technique for choosing points to minimize the interpolation error is extended to the closed interval $[0, T]$ by using the change of variable $t = \frac{T}{2}(\tilde{t} + 1)$ to transform the number $\tilde{t}_k$ in the interval $[-1, 1]$ into the corresponding number t_k in interval $[0, T]$. Taking into account equation (41), we form the similar functional,

$$S = \sum_{j=1}^{n} [w_j - \sum_{k=0}^{r-1} \beta_k P_k(t_j)]^2 + \alpha \sum_{j=1}^{n} (\sum_{k=0}^{r-1} \beta_k P_k(t_j))^2, \tag{47}$$

where

$$P_k(t_j) = \int_{s=0}^{t_j} T_k(s) k(t_j, s) ds, \quad j = 1, 2, \dots, n. \tag{48}$$

Differentiation of equation (47) with respect to β_i and setting the derivatives equal to zero, we obtain the set of linear equations

$$\sum_{k=0}^{r-1} \beta_k \sum_{j=i}^{n} [P_k(t_j)P_i(t_j) + \alpha T_k(t_j) T_i(t_j)] = \sum_{j=1}^{r-1} w_j P_i(t_j), i = 0, 1, \dots, r-1, \tag{49}$$

which is of the form $\mathbf{A}\beta = \mathbf{b}$, where $\beta = (\beta_0, \beta_1, \dots, \beta_{r-1})$, and $\mathbf{b} = (b_0, b_1, \dots, b_{r-1})$ where $b_i = \sum_{j=1}^{r-1} w_j P_i(t_j)$. The $r \times r$ coefficient matrix $\mathbf{A}$ has entries a_{kj}, $k, j = 0, 1, \dots, r-1$ given by

$$\begin{aligned} a_{kj} &= \sum_{j=1}^{n} [P_k(t_j)P_i(t_j) + \alpha T_k(t_j) T_i(t_j)], \\ i &= 0, 1, \dots, r-1, \quad k = 0, 1, \dots, r-1. \end{aligned} \tag{50}$$

After $\Delta t, \alpha$ and r are chosen ($n \geq r$), the set of r equations given by equation (50) is solved. To obtain reliable numerical results an initial value for $q(t)$ is known, namely, $q(0) = 0$, and can be used as a constraint. Embedded constraint have a similar effect on stability and accuracy [2].

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